

# RADIAL COMPONENTS, PREHOMOGENEOUS VECTOR SPACES, AND RATIONAL CHEREDNIK ALGEBRAS

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ABSTRACT. Let  $(\tilde{G} : V)$  be a finite dimensional representation of a connected reductive complex Lie group  $(\tilde{G} : V)$ . Denote by  $G$  the derived subgroup of  $\tilde{G}$  and assume that the categorical quotient  $V//G$  is one dimensional, i.e.  $\mathbb{C}[V]^G = \mathbb{C}[f]$  for a non constant polynomial  $f$ . In this situation there exists a homomorphism  $\text{rad} : \mathcal{D}(V)^G \rightarrow A_1(\mathbb{C})$ , the radial component map, where  $A_1(\mathbb{C})$  is the first Weyl algebra. We show that the image of  $\text{rad}$  is isomorphic to the spherical subalgebra of a rational Cherednik algebra whose multiplicity function is defined by the roots of the Bernstein-Sato polynomial of  $f$ . In the case where  $(\tilde{G} : V)$  is also multiplicity free we describe the kernel of  $\text{rad}$  and prove a Howe duality result between representations of  $G$  occurring in  $\mathbb{C}[V]$  and lowest weight modules over the Lie algebra generated by  $f$  and the “dual” differential operator  $\Delta \in S(V)$ ; this extends results of H. Rubenthaler obtained when  $(\tilde{G} : V)$  is a parabolic prehomogeneous vector space. If  $(\tilde{G} : V)$  satisfies a Capelli type condition, some applications are given to holonomic and equivariant  $D$ -modules on  $V$ . These applications are related to results proved by M. Muro or P. Nang in special cases of the representation  $(\tilde{G} : V)$ .

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## 1. INTRODUCTION

The base field is the field  $\mathbb{C}$  of complex numbers. Let  $(G : V)$  be a finite dimensional representation of a connected reductive Lie group  $G$ . The action of  $G$  extends to various algebras:  $\mathbb{C}[V] = S(V^*)$  the polynomial functions on  $V$ ,  $\mathcal{D}(V)$  the differential operators on  $V$  with coefficients in  $\mathbb{C}[V]$  and  $S(V)$  identified with differential operators on  $V$  with constant coefficients. Recall that  $\mathcal{D}(V) \cong \mathbb{C}[V] \otimes S(V)$  as a  $(\mathbb{C}[V], G)$ -module and that  $g \in G$  acts on  $D \in \mathcal{D}(V)$  by  $(g.D)(\varphi) = g.D(g^{-1}.\varphi)$  for all  $\varphi \in \mathbb{C}[V]$ . We thus obtain algebras of invariants  $\mathbb{C}[V]^G$ ,  $S(V)^G$  and  $\mathcal{D}(V)^G$ . Then  $\mathbb{C}[V]^G$  is (by definition) the algebra of regular functions on the categorical quotient  $V//G$  and one can define the algebra  $\mathcal{D}(V//G)$  of differential operators on this quotient (see [12] or [32]).

If  $D \in \mathcal{D}(V)^G$  and  $f \in \mathbb{C}[V]^G$  one obviously has  $D(f) \in \mathbb{C}[V]^G$ ; this gives an algebra homomorphism:

$$\mathcal{D}(V)^G \longrightarrow \mathcal{D}(V//G), \quad D \mapsto \{f \mapsto D(f), f \in \mathbb{C}[V]^G\}.$$

In general  $V//G$  is singular and  $\mathcal{D}(V//G)$  is difficult to describe. We will be interested here in the case where  $V//G$  is smooth, i.e. isomorphic to  $\mathbb{C}^\ell$  for some  $\ell \in \mathbb{N}$ , in which case  $\mathcal{D}(V//G)$  is isomorphic to the Weyl algebra  $A_\ell(\mathbb{C})$ . More precisely, we want to work with polar representations as defined by J. Dadoc and V. Kac in [5]. In this case there exists a Cartan subspace  $\mathfrak{h} \subset V$ , a finite subgroup  $W \subset \mathrm{GL}(\mathfrak{h})$  generated by complex reflections ( $W \simeq N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ ), such that the restriction map  $\psi : \mathbb{C}[V]^G \rightarrow \mathbb{C}[\mathfrak{h}]^W$ ,  $\psi(f) = f|_{\mathfrak{h}}$ , is an isomorphism. Thus  $\psi$  yields the isomorphism  $V//G \xrightarrow{\sim} \mathfrak{h}/W \equiv \mathbb{C}^\ell$  and, consequently, an isomorphism  $\mathcal{D}(V//G) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}/W) \equiv A_\ell(\mathbb{C})$ . Recall that among the polar representations one finds two important classes:

- the representations with a one dimensional quotient, i.e.  $\dim V//G = 1$ ;
- the class of “theta groups”.

In the latter case there exists a semisimple Lie algebra  $\mathfrak{s}$  and a  $\mathbb{Z}_m$ -grading  $\mathfrak{s} = \bigoplus_{i=0}^{m-1} \mathfrak{s}_i$  such that  $(G : V)$  identifies with the representation of the adjoint group of  $\mathfrak{s}_0$  acting on  $\mathfrak{s}_1$ . This generalizes the case of symmetric pairs  $(G : V) = (K : \mathfrak{p})$  where (with obvious notation)  $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$  is the decomposition associated to a complexified Cartan involution on  $\mathfrak{s}$ . Here  $\mathfrak{h} \subset \mathfrak{p}$  is a usual Cartan subspace and  $W$  is a Weyl group (cf. [15]).

Return to a general polar representation  $(G : V)$ . Combining the morphism  $\mathcal{D}(V)^G \rightarrow \mathcal{D}(V//G)$  with the isomorphism  $\mathcal{D}(V//G) \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}/W)$  we get the *radial component map*:

$$\mathrm{rad} : \mathcal{D}(V)^G \longrightarrow \mathcal{D}(\mathfrak{h}/W), \quad \mathrm{rad}(D)(f) = \psi(D(\psi^{-1}(f))), \quad f \in \mathbb{C}[\mathfrak{h}]^W.$$

The morphism  $\mathrm{rad}$  has proved to be useful in the representation theory of semisimple Lie algebras, or symmetric pairs  $(\mathfrak{s} : \mathfrak{k})$  as above, see, e.g. [15, 54, 29, 30]. Two obvious questions arise: describe the algebra  $R = \mathrm{Im}(\mathrm{rad}) \subset \mathcal{D}(\mathfrak{h}/W)$  and the ideal  $J = \mathrm{Ker}(\mathrm{rad}) \subset \mathcal{D}(V)^G$ . Some answers have been given in particular cases, see for example [30, 50], and it is expected that the algebra  $\mathcal{D}(V)^G/J$  has a representation theory similar to that of factors of enveloping algebras of semisimple Lie algebras (cf. [51]).

It is known that in the case  $(G : V) = (K : \mathfrak{p})$  of a symmetric pair, the subalgebra  $\mathrm{rad}(S(\mathfrak{p})^K)$  of  $R$  can be described via the introduction Dunkl operators [7, 14, 6, 53]. It is therefore natural to use rational Cherednik algebras [8, 9, 11] to describe  $R$ . Recall that to each complex reflection group  $(W : \mathfrak{h})$  is associated an algebra  $\mathcal{H}(k)$  where  $k$  is a “multiplicity function” on the set of reflecting hyperplanes in  $\mathfrak{h}$ . Denoting by  $\mathfrak{h}^{\mathrm{reg}}$  the complement of these hyperplanes,  $\mathcal{H}(k)$  is a subalgebra of the crossed product  $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) \rtimes \mathbb{C}W$  generated by  $\mathbb{C}[\mathfrak{h}]$ ,  $\mathbb{C}W$  and a

subalgebra  $\mathbb{C}[T_1 \dots, T_\ell] \cong S(\mathfrak{h})$  where each  $T_i$  is a (generalized) Dunkl operator, see §2.1 for details. If  $\mathbf{e} = \frac{1}{|W|} \sum_{w \in W} w \in \mathbb{C}W$  is the trivial idempotent,  $\mathbf{e}\mathcal{H}(k)\mathbf{e}$  is called the spherical subalgebra. Then one can show that there exists an injective homomorphism

$$\text{res} : \mathbf{e}\mathcal{H}(k)\mathbf{e} \longrightarrow \mathcal{D}(\mathfrak{h}/W)$$

and we obtain in this way a family  $U(k) = \text{res}(\mathbf{e}\mathcal{H}(k)\mathbf{e})$  of subalgebras of  $\mathcal{D}(\mathfrak{h}/W)$ . One would like to obtain information on  $R$  by answering the following question:

*Does there exist a multiplicity function  $k$  such that  $R = U(k)$ ?*

For instance, suppose that  $(G : V) = (K : \mathfrak{p})$  as above. The reflecting hyperplanes are then parametrised by elements of the reduced root system  $\mathbf{R}$  defined by  $(\mathfrak{s}, \mathfrak{h})$  and one defines a multiplicity function by:

$$k(\alpha) = \frac{1}{2}(\dim \mathfrak{s}^\alpha + \dim \mathfrak{s}^{2\alpha}), \quad \alpha \in \mathbf{R},$$

where  $\mathfrak{s}^\beta$  is the root space associated to the root  $\beta$ . For this choice of  $k$  one can prove [31]:

**Theorem** (L-Stafford). *One has  $R = \text{Im}(\text{rad}) = U(k) \cong \mathbf{e}\mathcal{H}(k)\mathbf{e}$ .*

Our aim in this work is to analyse a simpler case,  $G$  semisimple and  $\dim V // G = \dim \mathfrak{h} = 1$  (hence  $W \simeq \mathbb{Z}/n\mathbb{Z}$ ) and to give some applications of the radial component map in this situation. The function  $k$  is then given by  $n - 1$  complex parameters  $k_1, \dots, k_{n-1}$ , and  $R, U(k)$  are subalgebras of the first Weyl algebra  $\mathbb{C}[z, \partial_z]$ . The paper is organized as follows.

In §2 we recall general facts about Cherednik algebras and their spherical subalgebras in the one dimensional case. We show (Proposition 2.8) that  $U(k) = \tilde{U}/(\Omega)$  where  $\tilde{U}$  is an algebra similar to  $U(\mathfrak{sl}(2))$  (as defined in [52]) and  $\Omega$  is a generator of the centre of  $\tilde{U}$ . This says in particular that the representation of  $U(k)$  is well understood (and already known).

In the third section we assume that  $V$  is a representation of the reductive group  $\tilde{G}$ ,  $G$  is the derived group of  $\tilde{G}$  and  $\mathbb{C}[V]^G = \mathbb{C}[f]$  for a non constant  $f$ . Then it is known that:  $\tilde{G}$  acts on  $V$  with an open orbit, i.e.  $(\tilde{G} : V)$  is a prehomogeneous vector space (PHV),  $S(V)^G = \mathbb{C}[\Delta]$ ,  $\Delta(f^{s+1}) = b(s)f^s$  where  $b(s) = c(s+1)(s + \alpha_1) \cdots (s + \alpha_{n-1})$  is the (Bernstein-)Sato polynomial of  $f$ . Choosing  $k_i = \alpha_i - 1 + \frac{i}{n}$ ,  $1 \leq i \leq n - 1$ , we prove that  $R = U(k)$  (Theorem 3.9).

In section 4 we assume furthermore that the representation  $(\tilde{G} : V)$  is multiplicity free (MF). By [18] this is equivalent to the fact that  $\mathcal{D}(V)^{\tilde{G}} = \mathbb{C}[E_0, \dots, E_r]$  is a commutative polynomial ring. If  $\Theta$  is the Euler vector field on  $V$  one can find polynomials  $b_{E_i}(s)$  such that, if  $\Omega_i = E_i - b_{E_i}(\Theta)$ ,  $J = \sum_{i=0}^r \mathcal{D}(V)^G \Omega_i$  (Theorem 4.11). We then give a duality (of Howe type) between representations of  $G$  and lowest weight modules over the Lie algebra generated by  $f$  and  $\Delta$  (which is infinite dimensional when  $\deg f \geq 3$ ). This duality recovers, and extends, results obtained by H. Rubenthaler [49] when  $(\tilde{G} : V)$  is of commutative parabolic type.

In the last section we specialize further to the case where  $(\tilde{G} : V)$  is of ‘‘Capelli type’’, i.e.  $(\tilde{G} : V)$  is an irreducible MF representation such that  $\mathcal{D}(V)^{\tilde{G}}$  is equal to the image of the centre of  $U(\tilde{\mathfrak{g}})$  under the differential  $\tau : \tilde{\mathfrak{g}} \rightarrow \mathcal{D}(V)$  of the  $\tilde{G}$ -action. These representations have been studied in [18], they fall into eight cases (see Appendix A). It is not difficult to see that  $J = [\mathcal{D}(V)\tau(\mathfrak{g})]^G$  when  $(\tilde{G} : V)$  is of Capelli type (Proposition 5.3). We first apply this result to study  $\mathcal{D}_V$ -modules of the form  $\mathcal{M}(g, k) = \mathcal{D}(V)/(\mathcal{D}(V)\tau(\mathfrak{g}) + \mathcal{D}(V)q(\Theta)Q_k)$  where  $q(s)$  is a polynomial and  $Q_k = f^k$  or  $\Delta^k$ . We show in Theorem 5.9 that  $\mathcal{M}(g, k)$  is holonomic if and only if  $q(s) \neq 0$ . This has the well known consequence that the

space of hyperfunction solutions of  $\mathcal{M}(g, k)$  is finite dimensional. These properties generalize results obtained by M. Muro [36, 37]. For the second application, recall first the classical fact [21] that if  $(\tilde{G} : V)$  is MF, there is a finite number of  $\tilde{G}$ -orbits  $\mathcal{O}_i$ ,  $1 \leq i \leq t$ , in  $V$ . Let  $\tilde{\mathcal{C}} = \bigcup_{i=1}^t \overline{T_{\mathcal{O}_i}^* V}$  be the union of the conormal bundles to the orbits. P. Nang has shown that, when  $(\tilde{G} : V) = (\mathrm{SO}(n) \times \mathbb{C}^* : \mathbb{C}^n)$ ,  $(\mathrm{GL}(n) \times \mathrm{SL}(n) : \mathrm{M}_n(\mathbb{C}))$  or  $(\mathrm{GL}(2n) : \bigwedge^2 \mathbb{C}^{2n})$ , the category  $\mathrm{mod}_{\tilde{\mathcal{C}}}^{\mathrm{rh}}(\mathcal{D}_V)$  of regular holonomic  $\mathcal{D}_V$ -modules whose characteristic variety is contained in  $\tilde{\mathcal{C}}$  is equivalent to the category  $\mathrm{mod}^\theta(R)$  of finitely generated  $R$ -modules on which  $\theta = z\partial_z$  acts locally finitely. These representations are of Capelli type. We conjecture (see Conjecture 5.17) that when  $(\tilde{G} : V)$  is of Capelli type the category  $\mathrm{mod}^{G \times \mathbb{C}^*}(\mathcal{D}_V)$  of  $(G \times \mathbb{C}^*)$ -equivariant  $\mathcal{D}_V$ -modules is equivalent to  $\mathrm{mod}^\theta(R)$ . If  $G$  is simply connected,  $\mathrm{mod}^{G \times \mathbb{C}^*}(\mathcal{D}_V) = \mathrm{mod}_{\tilde{\mathcal{C}}}^{\mathrm{rh}}(\mathcal{D}_V)$  and the conjecture covers Nang's results; since  $\mathrm{mod}^\theta(R)$  can be easily described as a quiver category (i.e. finite diagrams of linear maps) its validity would give a simple classification of  $(G \times \mathbb{C}^*)$ -equivariant  $\mathcal{D}_V$ -modules. One can observe (Proposition 5.16) that, as in [39, 41, 43], the proof of the conjecture reduces to show that any  $M \in \mathrm{mod}^{G \times \mathbb{C}^*}(\mathcal{D}_V)$  is generated by its  $G$ -fixed points.

## 2. RATIONAL CHEREDNIK ALGEBRAS OF RANK ONE

**2.1. The spherical subalgebra and its restriction.** In this section we summarize some of the results we will need about rational Cherednik algebras. We begin with some general facts, see for example [9, 8, 11, 14].

Let  $\mathfrak{h}$  be a complex vector space of dimension  $\ell$  and  $W \subset \mathrm{GL}(\mathfrak{h})$  be an arbitrary complex reflection group. Denote by  $\mathcal{A} = \{H_s\}_{s \in \mathcal{S}}$  the collection of reflecting hyperplanes associated to  $W$  (where  $s \in \mathcal{S} \subset W$  is a complex reflection). Let  $\alpha_s \in \mathfrak{h}^*$  such that  $H_s = \alpha_s^{-1}(0)$  is the reflecting hyperplane associated to  $s \in \mathcal{S}$ . Fix  $H = H_s \in \mathcal{A}$ ; recall that the isotropy group  $W_H = \{w \in W : w|_H = \mathrm{id}_H\}$  is cyclic of order  $n_H$  (this order only depends on the conjugacy class of  $s$ ). Let  $e_{H,i} \in \mathbb{C}W_H$ ,  $0 \leq i \leq n_H - 1$ , be the primitive idempotents of  $\mathbb{C}W_H$ . Fix a family

$$k_{H_s,i} \in \mathbb{C}, \quad H_s \in \mathcal{A}, \quad 0 \leq i \leq n_{H_s} - 1, \quad k_{H_s,0} = 0,$$

of complex numbers such that  $k_{H_s,i} = k_{H_t,i}$  if  $s, t \in \mathcal{S}$  are conjugate. Such a family  $k = (k_{H,i})_{H,i}$  is called a multiplicity function. Let  $\mathfrak{h}^{\mathrm{reg}}$  be the complement of  $\bigcup_{s \in \mathcal{S}} H_s$  and set  $\pi = \prod_{s \in \mathcal{S}} \alpha_s$ . The group  $W$  acts naturally on  $\mathbb{C}[\mathfrak{h}] = S(\mathfrak{h}^*)$ ,  $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] = \mathbb{C}[\mathfrak{h}][\pi^{-1}]$ , hence on  $\mathrm{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]$  and  $\mathrm{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$ . These actions restrict to  $\mathcal{D}(\mathfrak{h})$  and  $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) = \mathcal{D}(\mathfrak{h})[\pi^{-1}]$ . Denote by  $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) \rtimes \mathbb{C}W$  the crossed product of the algebra  $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})$  by the group  $W$ . Recall that in that algebra we have:  $wfw^{-1} = w.f$ ,  $w\partial(y)w^{-1} = \partial(w.y)$  if  $f \in \mathbb{C}[\mathfrak{h}]$  and  $\partial(y)$  is the vector field defined by  $y \in V$ .

Then [8] one can introduce a subalgebra

$$\mathcal{H} = \mathcal{H}(W, k) \subset \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) \rtimes \mathbb{C}W$$

generated by three parts:  $\mathbb{C}[\mathfrak{h}]$ ,  $W$ ,  $\mathbb{C}[T(y) : y \in \mathfrak{h}] \cong S(\mathfrak{h})$ , where  $T(y)$  is a Dunkl operator defined as follows. Set  $a_{H_s}(k) = n_{H_s} \sum_{i=1}^{n_{H_s}-1} k_{H_s,i} e_{H_s,i} \in \mathbb{C}W_{H_s}$  and

$$T(y) = \partial(y) + \sum_{H_s \in \mathcal{A}} \frac{\langle \alpha_s, y \rangle}{\alpha_s} a_{H_s}(k) \in \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) \rtimes \mathbb{C}W.$$

Denote by  $\mathrm{res} : \mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) \rtimes \mathbb{C}W \rightarrow \mathrm{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$  the representation given by the natural action of  $W$  and  $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})$  on  $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$ . As observed in [8, §2.5] (see also [9, Proposition 4.5])  $\mathrm{res}(\mathcal{H}) \subset \mathrm{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]$  and this gives a natural structure of faithful

$\mathcal{H}$ -module on  $\mathbb{C}[\mathfrak{h}]$ , i.e. we have an injective homomorphism:

$$\text{res} : \mathcal{H} \longrightarrow \text{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}].$$

The group  $W$  acts on  $\mathcal{H}$ ,  $\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}W$  and  $\text{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]$  by conjugation, i.e  $w.u = wuw^{-1}$ . Denote by  $\mathcal{H}^W \subset (\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}W)^W$  and  $(\text{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}])^W$  the algebras of invariants under this action. Notice that if  $u \in \mathcal{H}$ ,  $w \in W$  and  $f \in \mathbb{C}[\mathfrak{h}]$ , we have:  $\text{res}(w.u)(f) = \text{res}(wuw^{-1})(f) = \text{res}(wu)(w^{-1}.f) = w.\text{res}(u)(w^{-1}.f) = (w.\text{res}(u))(f)$ . Thus the homomorphism  $\text{res}$  is  $W$ -equivariant and, in particular,  $\text{res} : \mathcal{H}^W \rightarrow (\text{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}])^W$ . Therefore  $w.\text{res}(u)(f) = w.\text{res}(u)(w^{-1}.f) = (w.\text{res}(u))(f) = \text{res}(u)(f)$ , for all  $u \in \mathcal{H}^W$ ,  $w \in W$ ,  $f \in \mathbb{C}[\mathfrak{h}]^W$ . We have obtained the following representation of  $\mathcal{H}^W$  on  $\mathbb{C}[\mathfrak{h}]^W$ :

$$\text{res} : \mathcal{H}^W \longrightarrow \text{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W.$$

(This morphism is not injective when  $W \neq \{1\}$ .) Let

$$\mathbf{e} = \frac{1}{|W|} \sum_{w \in W} w$$

be the trivial idempotent and define the *spherical subalgebra*:

$$\mathbf{e}\mathcal{H}\mathbf{e} = \mathbf{e}\mathcal{H}^W \subset \mathcal{H}^W. \quad (2.1)$$

Observe that  $\mathbf{e}\mathcal{H}\mathbf{e}$  is an algebra whose unit is equal to  $\mathbf{e}$ . From the previous discussion we obtain  $\mathbf{e}\mathcal{H}\mathbf{e} \subset \mathbf{e}(\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}W)\mathbf{e}$ . It is not difficult to show that  $\mathbf{e}(\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathbb{C}W)\mathbf{e} = \mathbf{e}\mathcal{D}(\mathfrak{h}^{\text{reg}})^W \cong \mathcal{D}(\mathfrak{h}^{\text{reg}})^W$ . It follows that  $u \in \mathbf{e}\mathcal{H}^W$  can be written  $u = ed$  for some  $d \in \mathcal{D}(\mathfrak{h}^{\text{reg}})^W$ , hence  $\text{res}(u)(f) = d(f)$  for all  $f \in \mathbb{C}[\mathfrak{h}]^W$ . This implies that  $\text{res}(u) \in \text{End}_{\mathbb{C}} \mathbb{C}[\mathfrak{h}]^W$  acts as the differential operator  $d$  on  $\mathbb{C}[\mathfrak{h}]^W$ . Consequently,  $\text{res}(u) \in \mathcal{D}(\mathfrak{h}/W) = \mathcal{D}(\mathbb{C}[\mathfrak{h}]^W) \subset \mathcal{D}(\mathfrak{h}^{\text{reg}}/W) \cong \mathcal{D}(\mathfrak{h}^{\text{reg}})^W$ . Furthermore it is easy to see that  $d = 0$  on  $\mathbb{C}[\mathfrak{h}]^W$  implies  $d = 0$ , hence  $u = 0$ . In conclusion: one has the injective restriction morphism (see [14] in the case of a Weyl group):

$$\text{res} : \mathbf{e}\mathcal{H}^W \longrightarrow \mathcal{D}(\mathfrak{h}/W), \quad \forall f \in \mathbb{C}[\mathfrak{h}]^W, D \in \mathcal{H}^W, \text{res}(\mathbf{e}D)(f) = \text{res}(D)(f). \quad (2.2)$$

We set:

$$U = U(W, k) = \text{res}(\mathbf{e}\mathcal{H}^W) \subset \mathcal{D}(\mathfrak{h}/W). \quad (2.3)$$

**2.2. The one dimensional case.** We now go the most simplest case of the previous construction: the case when  $\ell = \dim \mathfrak{h} = 1$ .

**Notation.** Let  $\mathfrak{h} = \mathbb{C}v$  be a one dimensional vector space and  $W \subset \text{GL}(\mathfrak{h})$  be a finite subgroup of order  $n$ . We adopt the following notation.

- $\mathbb{C}[\mathfrak{h}] = S(\mathfrak{h}^*) = \mathbb{C}[x]$ ,  $\langle x, v \rangle = 1$ ,  $\mathcal{D}(\mathfrak{h}) = \mathbb{C}[x, \partial_x]$ ;
- $W = \langle w \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ ,  $w.x = \zeta x$  where  $\zeta$  is a primitive  $n$ -th root of unity;
- $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[z]$ ,  $z = x^n$ ,  $\mathcal{D}(\mathfrak{h}/W) = \mathbb{C}[z, \partial_z]$ ,  $\theta = z\partial_z$ ;
- $\mathbf{e}_0 = \mathbf{e}, \mathbf{e}_1, \dots, \mathbf{e}_{n-1} \in \mathbb{C}W$  are the primitive idempotents (hence  $\mathbf{e}_i = \frac{1}{n} \sum_{j=0}^{n-1} \zeta^{ij} w^j$ );
- $k_0 = 0, k_1, \dots, k_{n-1} \in \mathbb{C}$ ;
- $T = T(v) = \partial_x + \frac{n}{x} \sum_{i=1}^{n-1} k_i \mathbf{e}_i \in \mathbb{C}[x^{\pm 1}, \partial_x] \rtimes \mathbb{C}W$ ;
- if  $p(s) \in \mathbb{C}[s]$  is a polynomial, set:  $\tau p(s) = p(s+1) - p(s)$ ,  $\tau^{j+1}p(s) = \tau(\tau^j p)(s)$ ,  $p^*(s) = p(s-1)$ .

The following well known lemma will prove useful (see [26] for a more general statement).

**Lemma 2.1.** *Let  $Q \in \mathbb{C}[z, \partial_z]$  satisfying:*

$$\exists p \in \mathbb{Z}, \quad \forall m \in \mathbb{N}, \quad Q(z^m) \in \mathbb{C}z^{m+p}.$$

*Then there exists a polynomial  $\varphi(s) \in \mathbb{C}[s]$  of degree  $d$  such that:  $Q$  has order  $d$  and can be written*

$$Q = z^p \varphi(\theta) = \sum_{j=0}^d q_j(z) \partial_z^j$$

where

$$q_j(z) = \frac{1}{j!} (\tau^j \varphi)(0) z^{j+p} \quad \text{and} \quad (\tau^j \varphi)(0) = 0 \quad \text{if } p + j < 0.$$

**Remark 2.2.** One can define the algebra  $\mathbb{C}[z^\alpha : \alpha \in \mathbb{Q}]$  by adjoining roots of polynomials of the form  $t^p - z$ ,  $p \in \mathbb{N}$  prime. The derivation  $\partial_z$  is naturally defined on this algebra by  $\partial_z(z^\alpha) = \alpha z^{\alpha-1}$ . Let  $Q$  be as in Lemma 2.1; then  $Q$  extends to  $\mathbb{C}[z^\alpha : \alpha \in \mathbb{Q}]$  by  $Q(z^\alpha) = \sum_j q_j(z) \partial_z(z^\alpha) = \varphi(\alpha) z^{\alpha+p}$ .

The next lemma is straightforward by direct computation.

**Lemma 2.3.** *The following formulas hold:*

- (a)  $[\mathbf{e}_i, x] = x(\mathbf{e}_{i+1} - \mathbf{e}_i)$  (where  $\mathbf{e}_n = \mathbf{e}_0 = \mathbf{e}$ );
- (b)  $[T, x] = 1 + n \sum_{i=1}^{n-1} k_i (\mathbf{e}_{i+1} - \mathbf{e}_i) = 1 + n \sum_{i=0}^{n-1} (k_i - k_{i+1}) \mathbf{e}_i$ ;
- (c)  $wTw^{-1} = \zeta^{-1}T$ ;
- (d) let  $p \in \mathbb{N}$  and define  $q \in \{0, \dots, n-1\}$  by  $p + q \equiv 0 \pmod{n}$ , then  $T(x^p) = (nk_q + p)x^{p-1}$ ;
- (e) let  $1 \leq j \leq n$  and  $s \in \mathbb{N}$ , then

$$T^j(x^{sn}) = \prod_{i=1}^j (nk_{i-1} + sn - i + 1) x^{sn-j},$$

in particular  $(T/n)^n(z^s) = \prod_{i=0}^{n-1} (s + k_i - i/n) z^{s-1}$ .

We now introduce the rational Cherednik algebra, and its spherical subalgebra, in the rank one case.

**Definition 2.4.** The rational Cherednik algebra associated to  $W$  with parameters  $k_i$ ,  $0 \leq i \leq n-1$ , is the subalgebra of  $\mathbb{C}[x^{\pm 1}, \partial_x] \rtimes \mathbb{C}W$  defined by:

$$\mathcal{H} = \mathcal{H}(W, k_0, \dots, k_{n-1}) = \mathbb{C}\langle x, T, w \rangle$$

Its spherical subalgebra is  $\mathbf{e}\mathcal{H}\mathbf{e}$ .

Observe that when  $n = 1$  (i.e.  $W$  trivial) the algebra  $\mathcal{H} = \mathbf{e}\mathcal{H}\mathbf{e}$  is nothing but  $\mathcal{D}(\mathfrak{h}) = \mathbb{C}[x, \partial_x]$  and all the results we are going to obtain are in this case obvious. We therefore will only be interested in the case  $n \geq 2$ .

It is easily seen that:

- $\mathbf{e}\mathcal{H}\mathbf{e} = \mathbf{e}\mathcal{H}^W = \mathbb{C}\langle \mathbf{e}, \mathbf{e}x^n, \mathbf{e}(T/n)^n, \mathbf{e}xT/n \rangle$ ;
- the image  $U = \text{res}(\mathbf{e}\mathcal{H}\mathbf{e})$  of the injective homomorphism, defined in (2.2),

$$\text{res} : \mathbf{e}\mathcal{H}^W \longrightarrow \mathcal{D}(\mathfrak{h}/W) = \mathbb{C}[z, \partial_z], \quad (2.4)$$

is generated by  $z$ ,  $\text{res}(\mathbf{e}(T/n)^n)$  and  $\text{res}(\mathbf{e}xT/n)$ .

- there exists a finite dimensional filtration on  $\mathbf{e}\mathcal{H}\mathbf{e}$  such that the associated graded algebra  $\text{gr}(\mathbf{e}\mathcal{H}\mathbf{e})$  is isomorphic to  $S(\mathfrak{h}^* \times \mathfrak{h})^W \equiv \mathbb{C}[X, Y, S]/(XY - S^n)$ , cf. [9, p. 262] (one has  $X \equiv \text{gr}(\mathbf{e}x^n)$ ,  $Y \equiv \text{gr}(\mathbf{e}T^n)$ ,  $S \equiv \text{gr}(\mathbf{e}xT)$ ).

Fix a constant  $c \in \mathbb{C}^*$  and set:

$$\lambda_i = k_i - \frac{i}{n}, \quad b^*(s) = c \prod_{i=0}^{n-1} (s + \lambda_i), \quad b(s) = b^*(s+1) = c \prod_{i=0}^{n-1} (s + \lambda_i + 1) \quad (2.5)$$

$$v(s) = -2b(-s), \quad \psi(s) = \frac{1}{2}(\tau v)(s) = b(-s) - b(-s-1). \quad (2.6)$$

**Proposition 2.5.** *Set  $\delta = c \operatorname{res}(\mathbf{e}(T/n)^n)$ . Then  $U = \operatorname{res}(\mathbf{e}\mathcal{H}\mathbf{e}) = \mathbb{C}[z, \theta, \delta]$  and one has:*

- (1)  $\delta = z^{-1}b^*(\theta) = \sum_{j=1}^n \frac{1}{j!}(\tau^j b^*)(0)z^{j-1}\partial_z^j$ ;
- (2)  $\operatorname{res}(\mathbf{e}xT/n) = \theta$ ;
- (3)  $[\delta, z] = \psi(-\theta) = b(\theta) - b(\theta-1) = (\tau b^*)(\theta)$ ;
- (4)  $[\theta, z] = z, [\theta, \delta] = -\delta$ ;
- (5)  $2z\delta + v(-\theta+1) = 2(z\delta - b^*(\theta)) = 0$ .

*Proof.* The equality  $U = \mathbb{C}[z, \theta, \delta]$  is clear.

(1) From the definition of the map  $\operatorname{res}$ , cf. (2.2), and Lemma 2.3(e) we deduce that  $\delta(z^s) = c(T/n)^n(z^s) = b^*(s)z^{s-1}$ . The claim therefore follows from Lemma 2.1 applied to  $Q = \delta$ ,  $\varphi = b^*$  and  $p = -1$ .

(2) By Lemma 2.3(e) again we get that  $\operatorname{res}(xT)(z^s) = xT(x^{sn}) = snx^{sn} = nsz^s$ , i.e.  $\operatorname{res}(\mathbf{e}xT/n) = \theta$ .

(3) Using (1) we obtain that  $[\delta, z](z^s) = (\tau b^*)(s)z^s = (\tau b^*)(\theta)(z^s)$ , hence  $[\delta, z] = (\tau b^*)(\theta)$ .

The formulas in (4) and (5) are obvious.  $\square$

**2.3. Algebras similar to  $U(\mathfrak{sl}(2))$ .** We recall here the definition, and some properties, of the algebras similar to  $U(\mathfrak{sl}(2))$  introduced in [20] and [52].

Let  $\psi(s) \in \mathbb{C}[s]$  be an arbitrary polynomial of degree  $\geq 1$  and write  $\psi = \frac{1}{2}\tau v$  for some  $v \in \mathbb{C}[s]$  of degree  $n \geq 2$ . Define a  $\mathbb{C}$ -algebra  $\tilde{U}$  by generators and relations as follows (cf. [52]):

$$\tilde{U} = \tilde{U}(\psi) = \mathbb{C}\langle A, B, H \rangle, \quad [A, B] - \psi(H) = 0, \quad [H, A] - A = 0, \quad [H, B] + B = 0.$$

Note that when  $\deg \psi = 1$ , i.e.  $n = 2$ , one has  $\tilde{U} = U(\mathfrak{sl}(2))$ . The algebra  $\tilde{U}$  has the following properties, see [52, 19, 38].

- The centre of  $\tilde{U}$  is  $Z(\tilde{U}) = \mathbb{C}[\Omega]$ ,  $\Omega = 2BA + v(H+1) = 2AB + v(H)$ .
- For  $\lambda \in \mathbb{C}$  one defines the “Verma module”  $M(\lambda) = \tilde{U} \otimes_{\mathbb{C}[H, A]} \mathbb{C}_\lambda$ , where  $\mathbb{C}_\lambda$  is the one dimensional module associated to  $\lambda$  over the solvable Lie algebra  $\mathbb{C}A + \mathbb{C}H$ .
- Each  $M(\lambda)$  has a unique simple quotient  $L(\lambda)$  and any finite dimensional  $\tilde{U}$ -module is of the form  $L(\lambda)$  for some  $\lambda$ .
- The primitive ideals of  $\tilde{U}$  are the annihilators  $\operatorname{ann} L(\lambda)$ ; the minimal primitive ideals are the  $\operatorname{ann} M(\lambda) = (\Omega - v(\lambda+1))$ , they are completely prime. If  $I$  is an ideal strictly containing  $\operatorname{ann} M(\lambda)$ , then  $\dim_{\mathbb{C}} \tilde{U}/I$  is finite.
- One can define in an obvious way a category  $\mathcal{O}$  for  $\tilde{U}$  which decomposes as:

$$\mathcal{O} = \bigsqcup_{\alpha} \mathcal{O}_{\alpha}, \quad \mathcal{O}_{\alpha} = \{M \in \mathcal{O} : (\Omega - \alpha)^k M = 0 \text{ for some } k\}.$$

Moreover  $\mathcal{O}_{\alpha} \equiv \operatorname{mod} A$  for a finite dimensional  $\mathbb{C}$ -algebra  $A$ .

The representation theory of the algebras  $\tilde{U}/(\Omega - v(\lambda+1))$  is therefore quite well understood.

We will be interested in the algebra  $U = \tilde{U}/(\Omega) = \mathbb{C}[a, b, h]$  where  $a, b, h$  are the classes of  $A, B, H$ . We have in  $U$ :

$$[a, b] = \psi(h) = \frac{1}{2}(\tau v)(h), \quad [h, a] = a, \quad [h, b] = -b, \quad 2ab = -v(h).$$

For simplicity we will assume that  $v(1) = 0$ . Recall then that  $M(0) \equiv \mathbb{C}[b]$  is a faithful  $U$ -module where  $h.b^k = -kb^k$ ,  $b.b^k = b^{k+1}$ ,  $a.b^k = (\sum_{i=1}^{k-1} \psi(-i))b^{k-1}$  for all  $k \geq 0$ . We want to study the Lie subalgebra  $\mathcal{L}$  of  $(U, [\ , \ ])$  generated by the elements  $a, b$ . Recall [52] that when  $n \leq 2$  this algebra is finite dimensional. The algebra  $\mathcal{L}$  acts on  $\mathbb{C}[b] \equiv M(0)$  and for each  $i \in \mathbb{Z}$  we set:

$$\mathcal{L}_i = \{u \in \mathcal{L} : u.b^k \in \mathbb{C}b^{k+i} \text{ for all } k \in \mathbb{N}\}.$$

Clearly:  $a \in \mathcal{L}_{-1}$ ,  $h \in \mathcal{L}_0$ ,  $b \in \mathcal{L}_1$ .

**Lemma 2.6.** (1) *The element  $h$  is transcendental over  $\mathbb{C}$ .*

(2) *For  $g(h) \in \mathbb{C}[h]$  we have  $[b, ag(h)] = -\frac{1}{2}\tau(vg^*)(h)$ .*

(3) *There exists a sequence  $(g_m(h))_{m \in \mathbb{N}} \subset \mathcal{L} \cap \mathbb{C}[h]$  such that  $\deg g_m = (m+1)n - (2m+1)$ . In particular,  $\deg g_m < \deg g_{m+1}$  when  $n \geq 3$ .*

*Proof.* (1) This follows, for example, from  $g(h).b^k = g(-k)b^k$  in  $M(0)$  for all  $g(h) \in \mathbb{C}[h] \subset U$ .

(2) Recall [52, Appendix] that  $[b, g(h)] = b(g(h) - g(h-1))$  and  $[g(h), a] = a\tau(g)(h)$ . Thus:

$$\begin{aligned} [b, ag(h)] &= [b, a]g(h) + a[b, g(h)] = -\frac{1}{2}(\tau v)(h)g(h) + ab(g(h) - g(h-1)) \\ &= -\frac{1}{2}(v(h+1) - v(h))g(h) - \frac{1}{2}v(h)(g(h) - g(h-1)) \\ &= -\frac{1}{2}(v(h+1)g(h) - v(h)g(h-1)) \\ &= -\frac{1}{2}(\tau(vg^*))(h), \end{aligned}$$

as desired.

(3) We start with  $g_0(h) = \psi(h) = [a, b] \in \mathcal{L}$ . Then,  $\deg g_0 = n-1$  and, by (2),  $[b, a(\tau g_0)(h)] = -\frac{1}{2}\tau(v(\tau g_0)^*)(h) \in \mathcal{L}$ . We thus set  $g_1(h) = -\frac{1}{2}\tau(v(\tau g_0)^*)(h)$ ; note that  $\deg \tau(v(\tau g_0)^*) = \deg v + \deg \tau g_0 - 1 = 2n-3$ . Suppose that  $g_m(h) \in \mathcal{L}$  has been obtained; then  $[g_m(h), a] = a(\tau g_m)(h) \in \mathcal{L}$  and  $[b, [g_m(h), a]] = -\frac{1}{2}\tau(v(\tau g_m)^*)(h) \in \mathcal{L}$ . Set  $g_{m+1}(h) = \tau(v(\tau g_m)^*)(h)$ . We have  $\deg g_{m+1} = \deg v + \deg g_m - 2 = (m+1)n - (2m+1) + n - 2 = (m+2)n - (2(m+1) + 1)$ , hence the result.  $\square$

**Proposition 2.7.** *Assume that  $n \geq 3$ . Then  $\dim_{\mathbb{C}} \mathcal{L}_i = \infty$  for all  $i \in \mathbb{Z}$ .*

*Proof.* When  $i = 0$  the claim follows from Lemma 2.6(1). Suppose that  $i > 0$ . Let  $(g_m(h))_m \subset \mathcal{L}_0$  be as in Lemma 2.6(3). We will now show that  $a^i(\tau^i g_m)(h) \in \mathcal{L}_{-i}$  for all  $m$ . Since  $\deg g_{m+1} > \deg g_m$  (because  $n \geq 3$ ) the elements  $a^i(\tau^i g_m)(h)$  are linearly independent in the domain  $U$ . We argue by induction on  $i$ . When  $i = 1$  the claim follows from  $[a, g_m(h)] = a(\tau g_m)(h) \in \mathcal{L}_{-1}$  for all  $m$ . Assume that  $a^i(\tau^i g_m)(h) \in \mathcal{L}_{-i}$  for all  $m$ , then:  $[a, a^i(\tau^i g_m)(h)] = a^{i+1}(\tau^{i+1} g_m)(h) \in \mathcal{L}_{-(i+1)}$  for all  $m$ .

Observe that there exists an anti-automorphism of  $\tilde{U}$  given by  $\varkappa(A) = B$ ,  $\varkappa(B) = A$ ,  $\varkappa(H) = H$ . It satisfies  $\varkappa(\Omega) = \Omega$ , therefore  $\varkappa$  induces an anti-automorphism of  $U$ . Since  $\varkappa(a^i(\tau^i g_m)(h)) = (\tau^i g_m)(h)b^i \in \mathcal{L}_i$ , we get that  $\dim \mathcal{L}_i = \infty$ .  $\square$

*Remark.* The fact that  $\dim_{\mathbb{C}} \mathcal{L} = \infty$  when  $n \geq 3$  can also be proved by using [48].

The next result shows that the spherical subalgebra  $e\mathcal{H}e$  is isomorphic to a quotient  $\tilde{U}/(\Omega)$  for an obvious choice of  $\psi(s)$ :

**Proposition 2.8.** *Let  $b^*(s), v(s) = -2b(-s), \psi(s) = \frac{1}{2}(\tau v)(s) \in \mathbb{C}[s]$  be as in (2.5) and (2.6), and denote by  $U = \text{res}(e\mathcal{H}e) = \mathbb{C}[z, \theta, \delta]$  the image of the spherical*

subalgebra under the restriction map. Let  $\tilde{U} = \mathbb{C}\langle A, B, H \rangle$  be the algebra similar to  $U(\mathfrak{sl}(2))$  defined by  $\psi$ . Then the morphism

$$\pi : \tilde{U} \longrightarrow U, \quad \pi(A) = \delta, \quad \pi(B) = z, \quad \pi(H) = h = -\theta$$

induces an isomorphism  $\pi : \tilde{U}/(\Omega) \xrightarrow{\sim} U$ . In particular, there exists an isomorphism  $\tilde{U}/(\Omega) \xrightarrow{\sim} \mathfrak{e}\mathcal{H}\mathfrak{e}$ .

*Proof.* The existence of the surjective homomorphism  $\pi$  clearly follows from (3) and (4) in Proposition 2.5. By Proposition 2.5(5),  $\text{Ker}(\pi)$  contains  $(\Omega)$ . Since  $\pi(\tilde{U}) = U$  is not finite dimensional, it follows from one of the properties of  $\tilde{U}$  that  $\text{Ker}(\pi) = (\Omega)$ .  $\square$

From results of Smith [52], Musson–Van den Bergh [38], et al., one can for instance deduce the following properties of the algebra  $U \cong \mathfrak{e}\mathcal{H}\mathfrak{e}$ :

- The Verma modules over  $U$  are the  $M(\lambda)$ 's such that  $v(\lambda + 1) = -2b^*(-\lambda) = 0$ , i.e.  $\lambda = 0, \lambda_1, \dots, \lambda_{n-1}$ , cf. (2.5).
- $\dim L(\lambda) < \infty \iff b^*(-\lambda) = b^*(-\lambda + j) = 0$  for some  $j \in \mathbb{N}^*$ .
- The global homological dimension of  $U$  is:

$$\begin{cases} \infty & \text{if } b^*(-s) \text{ has multiple roots;} \\ 2 & \text{if } b^*(-s) \text{ has no multiple root and two roots differing by some } j \in \mathbb{N}^*; \\ 1 & \text{otherwise.} \end{cases}$$

When a root  $-\lambda$  of the polynomial  $b^*(s)$  is a rational number one can use Remark 2.2 to realize the Verma module  $M(\lambda)$  under the form  $\mathbb{C}[z]z^{-\lambda}$ , on which  $z, \theta, \delta$  act as differential operators. This is for example the case for  $\lambda = 0$ , where we have  $M(0) = \mathbb{C}[z]$ .

*Examples.* (1) Take  $b(s) = (s+1)(s+3/2)\cdots(s+(n+1)/2)$ ,  $\lambda_i = i/2$ ,  $0 \leq i \leq (n-1)/2$ . Verma modules:  $M(\lambda_i) = \mathbb{C}[z]z^{-i/2}$  with the natural action of  $z, \delta, \theta$ . Irreducible finite dimensional modules:  $L(\lambda_i) = M(\lambda_i)/M(\lambda_i - 1)$ ,  $i \geq 2$ , of dimension 1. The global dimension of  $U$  is 2.

(2) Take  $b(s) = (s+1)(s+5)(s+9)$ ,  $\lambda_i = 4i$ ,  $i = 0, 1, 2$ . Verma modules:  $M(4i) = \mathbb{C}[z]z^{-4i}$ ,  $M(0) \subset M(4) \subset M(8)$  with quotients  $L(4i) = M(4i)/M(4(i-1))$  simple of dimension 4. The global dimension of  $U$  is 2.

(3) Take  $b(s) = (s+1)(s+n/2)$ ,  $n \geq 3$ ,  $\lambda_0 = 0$ ,  $\lambda_1 = \frac{n-2}{2}$ . Verma modules:  $M(0) = \mathbb{C}[z]$ ,  $M(\frac{n-2}{2}) = \mathbb{C}[z]z^{-\frac{n-2}{2}}$ . There are two cases:

- $n = 2k$ , then  $M(0) = \mathbb{C}[z] \subset M(k-1) = \mathbb{C}[z]z^{-(k-1)}$  with quotient  $L(k-1)$  simple of dimension  $k-1$ ; the global dimension of  $U$  is 2;
- $n = 2k+1$ , then  $M(0) = \mathbb{C}[z]$  and  $M(k-\frac{1}{2}) = \mathbb{C}[z]z^{-(k-\frac{1}{2})}$  are simple; the global dimension of  $U$  is 1.

(4) Take

$$\begin{aligned} b(s) &= (s+1)^8 [(s+2/3)(s+4/3)(s+3/4)(s+5/4)(s+5/6)(s+7/6)]^4 \\ &\quad \times [(s+7/10)(s+9/10)(s+11/10)(s+13/10)]^2, \end{aligned}$$

$\lambda_0 = 0$ ,  $\lambda_1 = -1/3$ ,  $\lambda_2 = 1/3$ ,  $\lambda_3 = -1/4$ ,  $\lambda_4 = 1/4$ ,  $\lambda_5 = -1/6$ ,  $\lambda_6 = 1/6$ ,  $\lambda_7 = -3/10$ ,  $\lambda_8 = -1/10$ ,  $\lambda_9 = 1/10$ ,  $\lambda_{10} = 3/10$ . Verma modules:  $M(\lambda_i) = \mathbb{C}[z]z^{-\lambda_i}$  which are simple  $U$ -modules. The global dimension of  $U$  is  $\infty$ .

### 3. REPRESENTATIONS WITH A ONE DIMENSIONAL QUOTIENT

**3.1. Prehomogeneous vector spaces.** Let  $\tilde{G}$  be a connected reductive complex algebraic group. We denote by  $G = (\tilde{G}, \tilde{G})$  its derived subgroup, which is a connected semisimple group. Recall that  $\tilde{G} = GC$  where  $C = Z(\tilde{G})^0$ , the connected component of the center of  $\tilde{G}$ , is a torus.

Let  $\tilde{\rho} : \tilde{G} \rightarrow \mathrm{GL}(V)$  be a finite dimensional representation of  $\tilde{G}$ . Recall that  $f \in \mathbb{C}[V] = S(V^*)$  is a relative invariant of  $(\tilde{G} : V)$  if there exists a rational character  $\chi \in \mathbf{X}(\tilde{G})$  such that  $g.f = \chi(g)f$  for all  $g \in \tilde{G}$ .

One says, see [25, Chapter 2], that  $(\tilde{G} : V) = (\tilde{G}, \tilde{\rho}, V)$  is a (reductive) *prehomogeneous vector space* (PHV) if  $\tilde{G}$  has a dense orbit in  $V$ . We denote the complement of the dense orbit by  $S$ , it is called the singular set of  $(\tilde{G} : V)$ . Then it is known [25, Theorem 2.9] that the one-codimensional irreducible components of  $V \setminus S$  are of the form  $\{f_i = 0\}$ ,  $1 \leq i \leq r$ , for some relative invariants  $f_i$ . The  $f_i$  are algebraically independent and are called the basic relative invariants of  $(\tilde{G} : V)$ ; any relative invariant  $f$  can be (up to a non-zero constant) written as  $\prod_{i=1}^r f_i^{m_i}$ . When the singular set is a hypersurface  $(\tilde{G} : V)$  is called regular, cf. [25, Theorem 2.28].

Let  $\tilde{\rho}^* : \tilde{G} \rightarrow \mathrm{GL}(V^*)$  be the contragredient representation. Then, see [25, Proposition 2.21],  $(\tilde{G} : V^*)$  is a PHV. Recall that  $S(V) = \mathbb{C}[V^*]$  can be identified with the algebra of constant coefficients differential operators on  $V$ . If  $\varphi \in \mathbb{C}[V^*]$  we denote by  $\varphi(\partial)$  the corresponding differential operator. If  $f \in \mathbb{C}[V]$  is a relative invariant of degree  $n$  and weight  $\chi \in \mathbf{X}(\tilde{G})$ , there exists a relative invariant  $f^* \in \mathbb{C}[V^*]$  of degree  $n$  and weight  $\chi^{-1}$ . The following result summarizes [25, Proposition 2.22] and [23].

**Theorem 3.1** (Sato-Bernstein-Kashiwara). *Under the above notation, set  $\Delta = f^*(\partial) \in S(V)$ . There exists  $b(s) \in \mathbb{R}[s]$  of degree  $n$  such that:*

- (1)  $b(s) = c \prod_{i=0}^{n-1} (s + \lambda_i + 1)$ ,  $c > 0$ ;
- (2)  $\Delta(f^{s+1}) = b(s)f^s$ ;
- (3)  $\lambda_i + 1 \in \mathbb{Q}_+^*$ ,  $0 \leq i \leq n-1$ ,  $\lambda_0 = 0$ .

The polynomial  $b(s)$  is called a *b-function* of  $f$ . Since the form of the operator  $\Delta = f^*(\partial)$  will be important in the proof of Theorem 3.9, we briefly indicate its expression in a particular coordinate system (see [25, p. 38]).

Denote by  $\bar{a}$  the complex conjugate of  $a \in \mathbb{C}$  and set  $|a|^2 = a\bar{a} \in \mathbb{R}_+$ . Let  $\tilde{K}$  be a maximal compact subgroup of  $\tilde{G}$ , so that  $\tilde{G} = \tilde{K} \exp(i\mathfrak{k})$  is the complexification of  $\tilde{K}$  (where  $\mathfrak{k} = \mathrm{Lie}(\tilde{K})$ ). Fix a  $\tilde{K}$ -invariant non-degenerate Hermitian form  $\kappa$  on  $V$  such that  $\kappa(\lambda v, \mu w) = \bar{\lambda}\mu\kappa(v, w)$ ,  $\lambda, \mu \in \mathbb{C}$ ,  $v, w \in V$ . We choose a  $\kappa$ -orthonormal basis  $\{e_i\}_{1 \leq i \leq N}$  on  $V$ , with dual basis  $\{z_i = e_i^*\}_i$ . Define

$$\phi : V \longrightarrow V^*, \quad \phi(v) = v^* = \kappa(v, -). \quad (3.1)$$

In coordinates we have:  $\phi(\sum_i v_i e_i) = \sum_i \bar{v}_i z_i$ . Then  $\phi$  is a bijective  $\mathbb{C}$ -antilinear map such that  $\phi(h.v) = h.\phi(v)$  for all  $h \in \tilde{K}$ . The inverse of  $\phi$ , also denoted by  $\phi$ , is given by  $\phi(v^*) = v$ , i.e.  $\phi(\sum_i a_i z_i) = \sum_i \bar{a}_i e_i$ , and it also satisfies  $\phi(h.v^*) = h.\phi(v^*)$ . We can extend  $\phi$  to  $S(V) = \mathbb{C}[V^*]$  and  $S(V^*) = \mathbb{C}[V]$  by multilinearity. Thus we get a bijective  $\tilde{K}$ -equivariant  $\mathbb{C}$ -antilinear morphism:

$$\phi : \mathbb{C}[V] \longrightarrow \mathbb{C}[V^*], \quad \phi(f) = f^*. \quad (3.2)$$

Now, if  $f$  is a relative invariant of  $(\tilde{G} : V)$  associated to  $\chi$ , we obtain:

$$h.f^* = h.\phi(f) = \phi(h.f) = \phi(\chi(h)f) = \overline{\chi(h)}\phi(f) = \chi(h)^{-1}f^*$$

for all  $h \in \tilde{K}$ , which shows that  $f^*$  is a relative invariant corresponding to  $\chi^{-1}$ .

The expression of  $\phi(f) = f^*$  in the chosen basis is given as follows. If  $\mathbf{i} = (i_1, \dots, i_N) \in \mathbb{N}^N$  we set  $|\mathbf{i}| = \sum_{j=1}^N i_j$ ,  $\mathbf{i}! = \prod_{j=1}^N i_j!$ ,  $z^{\mathbf{i}} = z_1^{i_1} \cdots z_N^{i_N}$ ,  $e^{\mathbf{i}} =$

$e_1^{i_1} \cdots e_N^{i_N}$  and  $\partial^{\mathbf{i}} = \partial_{z_1}^{i_1} \cdots \partial_{z_N}^{i_N}$ . Then, if the polynomial  $f$  is written

$$f = \sum_{\{\mathbf{i} \in \mathbb{N}^N : |\mathbf{i}|=n\}} a_{\mathbf{i}} z^{\mathbf{i}}, \quad a_{\mathbf{i}} \in \mathbb{C}, \quad (3.3)$$

one has

$$f^* = \sum_{\{\mathbf{i} \in \mathbb{N}^N : |\mathbf{i}|=n\}} \overline{a_{\mathbf{i}}} e^{\mathbf{i}} \quad (3.4)$$

and therefore:

$$\Delta = \sum_{\{\mathbf{i} \in \mathbb{N}^N : |\mathbf{i}|=n\}} \overline{a_{\mathbf{i}}} \partial^{\mathbf{i}}, \quad b(0) = \Delta(f) = c \prod_{i=0}^{n-1} (\lambda_i + 1) = \sum_{\mathbf{i} \in \mathbb{N}^N} \mathbf{i}! |a_{\mathbf{i}}|^2. \quad (3.5)$$

**Remarks 3.2.** 1) If  $K$  is a maximal compact subgroup of the semisimple group  $G$  we can embed  $K$  in a maximal compact subgroup  $\tilde{K}$  of  $\tilde{G}$ . Note that any relative invariant of  $(\tilde{G} : V)$  is a  $G$ -invariant, and that  $f$  is  $G$ -invariant if and only if it is  $K$ -invariant. The previous construction shows that there exist  $\tilde{K}$ -equivariant bijective  $\mathbb{C}$ -antilinear morphisms  $\phi : S(V) \rightarrow S(V^*)$  and  $\phi : S(V^*) \rightarrow S(V)$  such that  $\phi \circ \phi = \text{id}$ . In particular we have  $\phi(S(V^*)^K) = S(V)^K$ , hence  $\phi(S(V^*)^G) = S(V)^G$ .

2) Observe that  $\phi : V \rightarrow V^*$  is  $K$ -equivariant, but, in general, there is no  $G$ -module isomorphism of between  $V$  and  $V^*$ . For example, in the case  $(\tilde{G} : V) = (\text{GL}(n, \mathbb{C}), \bigwedge^2 \mathbb{C}^n)$  one has  $(\bigwedge^2 \mathbb{C}^n)^* \cong \bigwedge^{n-2} \mathbb{C}^n$  as a  $G$ -module, which is not isomorphic to  $\bigwedge^2 \mathbb{C}^n$  when  $n > 2$ .

**3.2. PHV of rank one.** Let  $\tilde{G} = GC$  be as above and  $(\tilde{G} : V)$  be a finite dimensional representation of  $\tilde{G}$ . In this subsection we make the following hypothesis:

**Hypothesis A.** *There exists  $f \in S^n(V^*)$  such that:  $f \notin \mathbb{C}[V]^{\tilde{G}}$  and  $\mathbb{C}[V]^G = \mathbb{C}[f]$ .*

*Remarks.* (1) Assume that  $G$  is a semisimple group and  $(G : V)$  is a finite dimensional representation of  $G$  such that  $\mathbb{C}[V]^G = \mathbb{C}[f]$ ,  $f \notin \mathbb{C}^*$ . Let the group  $\mathbb{C}^*$  act on  $V$  by homotheties. Then  $(G \times \mathbb{C}^* : V)$  satisfies the hypothesis A. Therefore one could assume without loss of generality that  $C = \mathbb{C}^*$ .

(2) Let  $f$  be as in the previous hypothesis. Then, since  $G$  is semisimple, the polynomial  $f$  is irreducible. Let  $g \in C$ , then  $g.f \in \mathbb{C}[V]^G = \mathbb{C}[f]$  and  $\deg(g.f) = \deg(f)$ , hence  $g.f = \chi(g)f$  for some  $\chi(g) \in \mathbb{C}^*$ . It follows that  $\chi \in X(\tilde{G})$ , i.e.  $f$  is a relative invariant of  $(\tilde{G} : V)$ ; note that  $\chi \neq 1$ , since  $f \notin \mathbb{C}[V]^{\tilde{G}}$ . Assume that  $f_1$  is another relative invariant of  $(\tilde{G} : V)$ . Then  $f_1 \in \mathbb{C}[V]^G$  is homogeneous and this implies that  $f_1 = \alpha f^m$  for some  $\alpha \in \mathbb{C}, m \in \mathbb{N}$ .

**Proposition 3.3.** *Let  $f$  be as in hypothesis A.*

(i) *Let  $\mathbb{C}(V)$  be the fraction field of  $\mathbb{C}[V]$ . Then one has:*

$$\mathbb{C}(V)^G = \mathbb{C}(f), \quad \mathbb{C}(V)^{\tilde{G}} = \mathbb{C}.$$

*In particular,  $(\tilde{G} : V)$  is a PHV.*

(ii) *Let  $f^*$  be the relative invariant obtained in (3.3), then  $\mathbb{C}[V^*]^G = \mathbb{C}[f^*]$ .*

(iii) *The representation  $(G : V)$  is polar.*

*Proof.* (i) The equality  $\mathbb{C}(V)^G = \mathbb{C}(f)$  follows from [46, Theorem 3.3]. By the remark (2) above, we know that  $g.f = \chi(g)f$  for all  $g \in \mathbb{C}$ . Observe that  $\mathbb{C}(V)^{\tilde{G}} = (\mathbb{C}(V)^G)^C = \mathbb{C}(f)^C$  and let  $\varphi = p(f)/q(f) \in \mathbb{C}(f)^C$  where  $p(f), q(f) \in \mathbb{C}[f]$  are relatively prime polynomials in  $f$ . One easily sees that  $p(f)$  and  $q(f)$  are relative invariants, thus  $p(f) = \alpha f^k$  and  $q(f) = \beta f^\ell$ ,  $\alpha, \beta \in \mathbb{C}$ . It follows that  $\chi^{k-\ell} = 1$ , hence  $k = \ell$  and  $\varphi \in \mathbb{C}$ . From [46, Corollary, p. 156] one gets that  $(\tilde{G} : V)$  is a PHV.

(ii) Adopt the notation of Remarks 3.2. The map  $\phi : \mathbb{C}[V] \rightarrow \mathbb{C}[V^*]$  defined in (3.2) yields a bijective  $\mathbb{C}$ -antilinear morphism  $\mathbb{C}[V]^G = \mathbb{C}[V]^K \rightarrow \mathbb{C}[V^*]^K = \mathbb{C}[V^*]^G$ . Thus  $\mathbb{C}[V^*]^G = \mathbb{C}[f^*]$ .

(iii) Choose  $v \in V$  regular semisimple, i.e.  $G.v$  closed and  $\dim G.v \geq \dim G.v'$  for all closed orbits  $G.v'$ . Set  $\mathfrak{h} = \mathbb{C}v$  and  $\mathfrak{g} = \text{Lie}(G)$ . Then, see [5], one easily deduces that  $\mathfrak{h} = \mathfrak{h}_v = \{x \in V : \mathfrak{g}.x \subset \mathfrak{g}.v\}$  is a Cartan subspace for the  $G$ -action on  $V$ .  $\square$

From now on, we assume that the hypothesis A is satisfied and we fix a Cartan subspace  $\mathfrak{h} = \mathbb{C}v$  for the  $G$ -action on  $V$ . We set

- $x = v^*$ , hence  $\mathbb{C}[\mathfrak{h}] = S(\mathfrak{h}^*) = \mathbb{C}[x]$ ;
- $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ .

By [5] we know that there exists an isomorphism  $V//G \cong \mathfrak{h}/W$  given by the restriction map  $\psi : \mathbb{C}[V]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$ ,  $\psi(p(f)) = p(f)|_{\mathfrak{h}}$ . Since  $f$  is homogeneous of degree  $n$ ,  $\psi(f) \in \mathbb{C}[x]$  is a scalar multiple of  $z = x^n$ . Therefore, multiplying  $v$  by a non zero constant, we may assume that

$$\psi(f) = x^n, \quad W \equiv \langle w \rangle \subset \text{GL}(\mathfrak{h})$$

where  $w$  acts on  $x$  by  $w.x = \zeta x$ ,  $\zeta$  primitive  $n$ -th root of unity. We can therefore adopt the notation of §2.2. In particular,  $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[z]$  where  $z = x^n = \psi(f)$ .

Let  $b(s) = c \prod_{i=0}^{n-1} (s + \lambda_i + 1) \in \mathbb{C}[s]$  be a  $b$ -function of the relative invariant  $f$  as in Theorem 3.1. We can then define the rational Cherednik algebra

$$\mathcal{H} = \mathcal{H}(W, k_0, \dots, k_{n-1}) = \mathbb{C}\langle x, T, w \rangle,$$

where the parameters  $k_i$  are given by  $k_i = \lambda_i + \frac{i}{n}$ , cf. (2.5). Recall that the image  $U = \text{res}(\mathfrak{e}\mathcal{H}\mathfrak{e}) \subset \mathbb{C}[z, \partial_z]$  of the spherical subalgebra is generated by  $z, \delta, \theta$ , where

$$\theta = z\partial_z, \quad \delta = z^{-1}b^*(\theta) = z^{-1}c \prod_{j=0}^{n-1} (\theta + \lambda_j),$$

see Proposition 2.5.

Recall from §1 that we have a radial component map:

$$\text{rad} : \mathcal{D}(V)^G \longrightarrow \mathcal{D}(\mathfrak{h}/W) = \mathbb{C}[z, \partial_z], \quad \text{rad}(D)(p(z)) = \psi(D(p(f))),$$

for all  $p(z) \in \mathbb{C}[\mathfrak{h}]^W$ . The algebra of radial components is defined to be

$$R = \text{rad}(\mathcal{D}(V)^G) \subset \mathbb{C}[z, \partial_z]. \quad (3.6)$$

The aim of this section is to prove that  $R = U$ , see Theorem 3.9.

Before entering the proof, let us give some standard examples. A complete list of pairs  $(\tilde{G} : V)$  as above with  $V$  irreducible can be found in [25]. Recall that  $\Delta = f^*(\partial) \in S(V)$  is the differential operator constructed in (3.5).

*Examples.* (1)  $(\tilde{G} = \text{GL}(n) : V = S^2\mathbb{C}^n)$ ,  $W = \mathbb{Z}_n$ ,  $f = \det(x_{ij})$ ,  $\Delta = \det(\partial_{x_{ij}})$ ,  $b(s) = \prod_{i=0}^{n-1} (s + i/2 + 1)$ .

(2)  $(\tilde{G} = \text{E}_6 \times \mathbb{C}^* : V = \mathbb{C}^{27})$ ,  $W = \mathbb{Z}_3$ ,  $f = \text{cubic form}$ ,  $b(s) = (s+1)(s+5)(s+9)$ .

(3)  $(\tilde{G} = \text{SO}(n) \times \mathbb{C}^* : V = \mathbb{C}^n)$ ,  $W = \mathbb{Z}_2$ ,  $f = \text{quadratic form}$ ,  $\Delta = \text{Laplacian}$ ,  $b(s) = (s+1)(s+n/2)$ .

(4)  $(\tilde{G} = \text{SL}(5) \times \text{GL}(4) : V = \bigwedge^2 \mathbb{C}^5 \otimes \mathbb{C}^4)$ ,  $W = \mathbb{Z}_{40}$ ,  $\deg f = 40$ ,  $b(s)$  is:

$$(s+1)^8 [(s+2/3)(s+4/3)(s+3/4)(s+5/4)(s+5/6)(s+7/6)]^4 \\ [(s+7/10)(s+9/10)(s+11/10)(s+13/10)]^2.$$

(5)  $(\tilde{G} = \text{GL}(n) \times \text{SL}(n) : V = \text{M}_n(\mathbb{C}))$ ,  $W = \mathbb{Z}_n$ ,  $f = \det(x_{ij})$ ,  $\Delta = \det(\partial_{x_{ij}})$ ,  $b(s) = \prod_{i=0}^{n-1} (s + i + 1)$ .

(6)  $(\tilde{G} = \mathrm{Sp}(n) \times \mathrm{SO}(3) \times \mathbb{C}^* : V = \mathrm{M}_{2n,3}(\mathbb{C}))$ ,  $W = \mathbb{Z}_4$ ,  $\deg f = 4$ ,  $b(s) = (s+1)(s+3/2)(s+n)(s+n+1/2)$ .

Remark: The first five examples are regular irreducible PHV, but (6) gives an example of an irreducible PHV which is not regular [25]. The description of the Verma modules on  $U$  associated to examples (1) to (4) are given in §2.3.

Let  $\Theta$  be the Euler vector field on  $V$ ; thus  $\Theta(p) = np$  for all  $p \in S^n(V^*)$ . In particular  $\Theta(f) = nf$ , which implies that  $\mathrm{rad}(\Theta) = n\theta$ . Set

$$\bar{\Theta} = \frac{1}{n}\Theta, \quad (3.7)$$

so that  $\mathrm{rad}(\bar{\Theta}) = \theta$ .

**Lemma 3.4.** *One has:*

$$U \subset R, \quad U[z^{-1}] = R[z^{-1}] = \mathbb{C}[z^{\pm 1}, \partial_z].$$

*Proof.* Let  $\Delta = f^*(\partial) \in S(V)$  be as in in (3.5). By definition and Theorem 3.1 we obtain:  $\mathrm{rad}(\Delta)(z^{s+1}) = \psi(\Delta(f^{s+1})) = b(s)z^s$ . Hence  $\mathrm{rad}(\Delta) = \delta$ , see Proposition 2.5. From  $\mathrm{rad}(f) = z$  and  $\mathrm{rad}(\bar{\Theta}) = \theta$  it follows that  $U = \mathbb{C}[z, \delta, \theta] \subset R \subset \mathbb{C}[z, \partial_z]$ . Observe that  $U[z^{-1}] = \mathbb{C}[z^{\pm 1}, \delta, z^{-1}\theta = \partial_z] = \mathbb{C}[z^{\pm 1}, \partial_z] \subset R[z^{-1}] \subset \mathbb{C}[z^{\pm 1}, \partial_z]$ , thus  $U[z^{-1}] = R[z^{-1}] = \mathbb{C}[z^{\pm 1}, \partial_z]$ .  $\square$

Recall that  $\phi : \mathbb{C}[V] \rightarrow \mathbb{C}[V^*]$ ,  $\phi(p) = p^*$ , is the bijective  $\tilde{K}$ -equivariant  $\mathbb{C}$ -antilinear morphism defined in (3.2).

**Lemma 3.5.** *The map  $\phi$  extends to a  $\tilde{K}$ -equivariant  $\mathbb{C}$ -antilinear anti-automorphism of  $\mathcal{D}(V)$ , given by  $\phi(p) = \partial(p^*)$ . One has:*

$$\phi(\mathcal{D}(V)^G) = \mathcal{D}(V)^G, \quad \phi^2 = \mathrm{id}, \quad \phi(f) = \Delta = \partial(f^*), \quad \phi(\bar{\Theta}) = \bar{\Theta}.$$

*Proof.* In the coordinate system  $\{z_i, \partial_{z_i}\}_{1 \leq i \leq N}$  we have:  $\phi(z_i) = \partial_{z_i}$ ,  $\phi(\partial_{z_i}) = z_i$ ,  $\phi(a) = \bar{a}$  for  $a \in \mathbb{C}$ . Since  $\mathcal{D}(V) = \mathbb{C}[z_i, \partial_{z_j} : 1 \leq i, j \leq N]$  with relations  $[\partial_{z_j}, z_i] = \delta_{ij}$ , it is clear that  $\phi$  extends to a  $\mathbb{C}$ -antilinear anti-automorphism of  $\mathcal{D}(V)$  such that  $\phi^2 = \mathrm{id}$ . By construction  $\phi$  is  $\tilde{K}$ -equivariant, in particular  $K$ -equivariant if  $K \subset \tilde{K}$  is a maximal compact subgroup of  $G$ . From  $G = K_{\mathbb{C}} = K \exp(i\mathfrak{k})$  it follows that  $\mathcal{D}(V)^K = \mathcal{D}(V)^G$ , hence  $\phi(\mathcal{D}(V)^G) = \mathcal{D}(V)^G$ . The equality  $\phi(f) = \Delta$  is obvious and  $\phi(\bar{\Theta}) = \bar{\Theta}$  is consequence of  $\Theta = \sum_i z_i \partial_{z_i}$ .  $\square$

Recall that  $\mathrm{rad} : \mathcal{D}(V)^G \rightarrow R$ ; we now want to check that the anti-automorphism  $\phi$  induces an anti-automorphism on  $R$  such that  $\phi(z) = \delta$ . Denote by  $J$  the kernel of  $\mathrm{rad}$ , thus:

$$J = \{D \in \mathcal{D}(V)^G : D(f^m) = 0 \text{ for all } m \in \mathbb{N}\}.$$

Set

$$\mathbb{D} = \mathcal{D}(V)^G \supset \tilde{\mathbb{D}} = \mathcal{D}(V)^{\tilde{G}}. \quad (3.8)$$

Since  $\Theta \in \mathcal{D}(V)^G$  we can decompose  $\mathbb{D}$  under the adjoint action of  $\Theta$ :

$$\mathbb{D} = \bigoplus_{p \in \mathbb{Z}} \mathbb{D}[p], \quad \mathbb{D}[p] = \{D \in \mathbb{D} : [\Theta, D] = pD\}. \quad (3.9)$$

**Proposition 3.6.** *One has  $\phi(J) = J$ .*

*Proof.* As  $\phi^2 = \mathrm{id}$  we need to show that  $\phi(J) \subset J$ . Since  $J$  is an ideal of  $\mathcal{D}(V)^G$  it decomposes under the adjoint action of  $\Theta$ :  $J = \bigoplus_{p \in \mathbb{Z}} J[p]$ ,  $J[p] = J \cap \mathbb{D}[p]$ . Thus we only need to check that  $\phi(J[p]) \subset J$ . In the previous coordinate system  $\{z_i, \partial_{z_i}\}_{1 \leq i \leq N}$  we have:

$$\mathbb{D}[p] = \sum_{|i|-|j|=p} \mathbb{C} z^i \partial^j.$$

We can write  $D \in \mathbb{D}[p]$  in a unique way under the form

$$D = D_0 + \cdots + D_t, \quad D_k = \sum_{|j|=k-p} \left( \sum_{|i|=k} a_{i,j} z^i \right) \partial^j,$$

(thus  $D_k = 0$  when  $k < p$ ). If  $D_t \neq 0$ , we set  $t = \deg_z D$ .

Let  $D = \sum_{|i|-|j|=p} a_{i,j} z^i \partial^j$  be in  $J[p] \setminus \{0\}$ . As  $G$  acts linearly on  $V$  we have  $G.D_k \subset S^k(V^*)S^{k-p}(V)$ . Since  $D$  is  $G$ -invariant it follows that each  $D_k$  is  $G$ -invariant. We have:

$$\phi(D) = \sum_k \phi(D_k), \quad \phi(D_k) = \sum_{\{|i|=k, |j|=k-p\}} \overline{a_{i,j}} z^j \partial^i.$$

From the previous expression we get that  $\phi(D_k)(1) = 0$  when  $k > 0$ , hence

$$\phi(D)(1) = \phi(D_0)(1) = \phi(D_0) = \sum_{|j|=-p} \overline{a_{0,j}} z^j.$$

Assume that  $D_0 = \sum_{|j|=-p} a_{0,j} \partial^j \in S^p(V)^G$  is non zero. From Proposition 3.3(ii) we can deduce that  $p = -\ell n$ ,  $\ell \geq 0$ ,  $D_0 = \alpha \Delta^\ell$ ,  $\alpha \neq 0$ . By hypothesis  $D(f^\ell) = \sum_k D_k(f^\ell) = 0$ . Note that, since  $|j| = k + n\ell > n\ell$  implies  $\partial^j(f^\ell) = 0$ , we have  $D_k(f^\ell) = \sum_{\{|i|=k, |j|=k+n\ell\}} a_{i,j} z^i \partial^j(f^\ell) = 0$  if  $k > 0$ . Thus  $D_0(f^\ell) = D(f^\ell) = 0$ . If  $\ell = 0$  we get  $D_0 = \alpha = D_0(1) = D_0(f^\ell) = 0$ , contradiction. Therefore  $\ell \geq 1$  and

$$0 = D(f^\ell) = D_0(f^\ell) = \alpha \Delta^\ell(f^\ell) = \alpha b(\ell - 1) \cdots b(0).$$

It is easily seen that  $\Delta^\ell(f^\ell) \neq 0$ , see [25, Proof of Proposition 2.22] (this is equivalent to  $b(j) \neq 0$  for all  $j \in \mathbb{N}$ ), hence a contradiction. Thus:  $\phi(D)(1) = \phi(D_0) = D_0 = 0$ .

We show that  $\phi(D) \in J$  by induction on  $t = \deg_z D$ . (In the case  $t = 0$  one has  $D = D_0 = 0$ .) Since  $\Delta \in S(V)^G$  and  $J$  is an ideal, one has  $[D, \Delta] \in J$ . Observe that

$$[D, \Delta] = \sum_k [D_k, \Delta], \quad [D_k, \Delta] = \sum_{|j|=k-p} \left( \sum_{|i|=k} a_{i,j} [z^i, \Delta] \right) \partial^j.$$

But  $\Delta \in S(V)^G$  implies that  $\deg_z [z^i, \Delta] < k = |i|$ , hence  $\deg_z [D, \Delta] < t = \deg_z D$ . Then, by induction,  $\phi([D, \Delta]) = [\phi(\Delta), \phi(D)] = [f, \phi(D)] \in J$ . If  $m \geq 0$  we then have:  $0 = [f, \phi(D)](f^m) = f\phi(D)(f^m) - \phi(D)(f^{m+1})$ . This implies, by induction on  $m$ ,  $\phi(D)(f^{m+1}) = f\phi(D)(f^m) = f^{m+1}\phi(D)(1)$ . It follows from the previous paragraph that  $\phi(D)(f^{m+1}) = \phi(D)(1) = 0$ , i.e.  $\phi(D) \in J$ .  $\square$

**Corollary 3.7.** (1) *There exists a  $\mathbb{C}$ -antilinear anti-automorphism  $\phi : R \rightarrow R$  such that:*

$$\phi^2 = \text{id}, \quad \phi(z) = \delta, \quad \phi(\theta) = \theta, \quad \phi(U) = U.$$

(2) *One has  $U[\delta^{-1}] = R[\delta^{-1}]$ .*

*Proof.* (1) Let  $\phi : \mathcal{D}(V)^G \rightarrow \mathcal{D}(V)^G$  be as in Lemma 3.5. By Proposition 3.6 we can define  $\phi : R \rightarrow R$  by setting

$$\phi(\text{rad}(D)) = \text{rad}(\phi(D)).$$

Indeed: if  $\text{rad}(D) = \text{rad}(D')$  we get  $D - D' \in J = \text{Ker}(\text{rad})$ , hence  $\phi(D) - \phi(D') \in J$  and  $\text{rad}(\phi(D)) = \text{rad}(\phi(D'))$ . The equality  $\phi^2 = \text{id}$  is clear; by definition and Lemma 3.5:

$$\phi(z) = \text{rad}(\phi(f)) = \text{rad}(\Delta) = \delta, \quad \phi(\theta) = \text{rad}(\phi(\bar{\Theta})) = \text{rad}(\bar{\Theta}) = \theta.$$

From  $U = \mathbb{C}[z, \delta, \theta]$  we then deduce  $\phi(U) = U$ .

(2) Observe that  $\text{ad}(\phi(u))^m(r) = (-1)^m \phi(\text{ad}(u)^m(r))$  for all  $u, r \in R$ . Since  $\text{ad}(z)$  is a locally nilpotent operator in  $R$ , it follows that  $\text{ad}(\phi(z)) = \text{ad}(\delta)$  has the same property. We can therefore construct the  $\mathbb{C}$ -algebras  $U[\delta^{-1}] \subset R[\delta^{-1}]$ .

Let  $Q = \text{Frac}(U)$  be the fraction field of the Noetherian domain  $U$ . By Lemma 3.4 we know that  $Q = \mathbb{C}(z, \partial_z) = \text{Frac}(R)$ . It is easy to see that  $\phi$  extends to  $Q$  by  $\phi(s^{-1}a) = \phi(a)\phi(s)^{-1}$  for all  $a \in R$ ,  $0 \neq s \in R$ . This gives a  $\mathbb{C}$ -antilinear anti-automorphism of  $Q$ . Then  $\phi(R[z^{-1}]) = \phi(U[z^{-1}]) = \phi(U)[\delta^{-1}] = U[\delta^{-1}]$ , which yields  $U[\delta^{-1}] = \phi(R[z^{-1}]) = \phi(R)[\delta^{-1}] = R[\delta^{-1}]$ , as desired.  $\square$

Let  $M$  be a module over a  $\mathbb{C}$ -algebra  $A$ , then the Gelfand-Kirillov of  $M$  is denoted by  $\text{GKdim}_A M$  or simply  $\text{GKdim } M$ , see [32].

**Lemma 3.8.** *Let  $r \in R$ . Then:*

$$\text{GKdim}_U(U + Ur)/U \leq \text{GKdim}_U U - 2 = 0.$$

*Proof.* From Corollary 3.7 we deduce that there exists  $\nu \in \mathbb{N}$  such that  $z^\nu r \in U$  and  $\delta^\nu r \in U$ . Therefore the  $U$ -module  $(U + Ur)/U$  is a factor of  $U/(Uz^\nu + U\delta^\nu)$ . There exists on  $U \cong \tilde{U}/(\Omega)$  (cf. Proposition 2.8) a finite dimensional filtration such that  $\text{gr}(U)$  is isomorphic to the commutative algebra  $\mathbb{C}[X, Y, S]/(XY - S^n)$ , see §2.2 or [52, 38], where  $\text{gr}(z) = X$ ,  $\text{gr}(\delta) = Y$ . It follows that the associated graded module of  $U/(Uz^\nu + U\delta^\nu)$  is a factor of  $\text{gr}(U)/(\text{gr}(U)X^\nu + \text{gr}(U)Y^\nu)$ , which is finite dimensional. Hence the result.  $\square$

We now can prove the main result of this section.

**Theorem 3.9.** *One has  $U = R$ .*

*Proof.* Endow  $U$  with a filtration such that  $\text{gr}(U) \cong \mathbb{C}[X, Y, S]/(XY - S^n)$  as in the proof of the previous lemma. Observe that  $\mathbb{C}[X, Y, S]/(XY - S^n)$  is commutative Gorenstein normal domain. By [3, Theorem 3.9]  $U$  is Auslander-Gorenstein and by [55]  $U$  is a maximal order. Recall that  $Q = \text{Frac}(U)$  and consider the following family of finitely generated  $U$ -modules  $M$ :

$$\mathcal{F} = \{U \subset M \subset Q : \text{GKdim}_U M/U \leq \text{GKdim}_U U - 2\}.$$

From [3, Theorem 1.14] we know that  $\mathcal{F}$  contains a unique maximal element  $\tilde{M}$ . By Lemma 3.8 we have  $U + Ur \subset \tilde{M}$  for all  $r \in R$ ; hence  $R \subset \tilde{M}$ . It follows that  $R$  is finitely generated over  $U$  with  $Q = \text{Frac}(U) = \text{Frac}(R)$ . Thus  $U = R$ , since  $U$  is a maximal order.  $\square$

**Remark 3.10.** Let  $(G : V)$  be a representation of the connected reductive group  $G$  such that  $\dim V // G = 1$ . If  $\mathbb{C}[V]^G = \mathbb{C}[f]$  one can define  $\Delta \in S(V)^G$  and the polynomial  $b(s) = c \prod_{i=0}^{n-1} (s + \lambda_i + 1)$  as in Theorem 3.1. Then, the proof of Theorem 3.9 can be repeated to show that  $R = \text{Im}(\text{rad}) = U(k)$  (where  $k_i = \lambda_i + \frac{i}{n}$ ,  $0 \leq i \leq n-1$ ).

#### 4. MULTIPLICITY FREE REPRESENTATIONS

**4.1. Generalities.** Let  $(\tilde{G} : V)$  be a connected reductive group. Write  $\tilde{G} = GC$ ,  $C \cong (\mathbb{C}^*)^c$ , as in §3.1. We adopt the following notation:

- the Lie algebra of an algebraic group is denoted by the corresponding gothic character;
- $TU$  is a Borel subgroup of  $G$ ,  $T$  being a maximal torus of  $G$ , hence  $\tilde{T}U$  is a Borel subgroup of  $\tilde{G}$ ,  $\tilde{T} = TC$ ;
- $\mathbf{R}$  is the root system of  $(\mathfrak{g}, \mathfrak{t})$ ,  $\mathbf{B} = \{\alpha_1, \dots, \alpha_\ell\}$  is a basis of  $\mathbf{R}$  and  $\mathbf{R}^+$  is the set of associated positive roots;
- $\Lambda$  is the weight lattice of  $(\mathfrak{g}, \mathfrak{t})$ , thus  $\Lambda = \mathbb{Z}\varpi_1 \oplus \dots \oplus \mathbb{Z}\varpi_\ell$  where  $\langle \varpi_i, \alpha_j \rangle = \delta_{ij}$ ;  $\Lambda^+ = \mathbb{N}\varpi_1 \oplus \dots \oplus \mathbb{N}\varpi_\ell$  denotes the dominant weights;
- $\tilde{\Lambda} = \Lambda \oplus \mathbf{X}(C)$ , with  $\mathbf{X}(C) \cong \mathbb{Z}^c$ ;  $\tilde{\Lambda}^+ = \Lambda^+ \oplus \mathbf{X}(C)$ ;

- if  $\tilde{\lambda} \in \tilde{\Lambda}^+$ , resp.  $\lambda \in \Lambda^+$ , we denote by  $E(\tilde{\lambda})$ , resp.  $E(\lambda)$ , an irreducible  $\tilde{\mathfrak{g}}$ -module, resp.  $\mathfrak{g}$ -module, with highest weight  $\tilde{\lambda}$ , resp.  $\lambda$ ; the dual of  $E(\tilde{\lambda})$  is isomorphic to  $E(\tilde{\lambda}^*)$ ,  $\tilde{\lambda}^* = -w_0(\tilde{\lambda})$  where  $w_0$  is the longest element of the Weyl group of  $\mathfrak{R}$  (similarly for  $E(\lambda)^*$ ).

We fix a finite dimensional representation  $(\tilde{G} : V)$  of the reductive group  $\tilde{G}$ . Then the rational  $\tilde{G}$ -module  $\mathbb{C}[V] = S(V^*)$  decomposes as

$$\mathbb{C}[V] \cong \bigoplus_{\tilde{\lambda} \in \tilde{\Lambda}^+} E(\tilde{\lambda})^{m(\tilde{\lambda})}$$

where  $m(\tilde{\lambda}) \in \mathbb{N} \cup \{\infty\}$ .

**Definition 4.1.** The representation  $(\tilde{G} : V)$  is called multiplicity free (MF for short) if  $m(\tilde{\lambda}) \leq 1$  for all  $\tilde{\lambda}$ . In this case

$$\mathbb{C}[V] = \bigoplus_{\tilde{\lambda} \in \tilde{\Lambda}^+} V(\tilde{\lambda})^{m(\tilde{\lambda})}, \quad m(\tilde{\lambda}) = 0, 1$$

where  $V(\tilde{\lambda}) \subset S^{d(\tilde{\lambda})}(V^*)$  is isomorphic to  $E(\tilde{\lambda})$ ; if  $m(\tilde{\lambda}) = 1$ ,  $d(\tilde{\lambda})$  is called the degree of  $\tilde{\lambda}$  in  $\mathbb{C}[V]$ .

*Remark.* The MF representations are classified [21, 1, 27]. We give in Appendix A the list of  $(\tilde{G} : V)$  with  $V$  irreducible (see [21]). For instance, the examples (1), (2), (3), (5) given in §3.2 are MF.

From now on, let  $(\tilde{G} : V)$  be a MF representation. The following results can be found, for example, in [1, 18, 21, 26].

– Set  $\tilde{\Gamma} = \{\tilde{\lambda} : m(\tilde{\lambda}) = 1\}$ , then  $\tilde{\Gamma} = \bigoplus_{i=0}^r \mathbb{N}\tilde{\lambda}_i$  where the  $\tilde{\lambda}_i$  are linearly independent over  $\mathbb{Q}$ .

– The algebra of  $U$ -invariants  $\mathbb{C}[V]^U = \mathbb{C}[h_0, \dots, h_r]$  is a polynomial ring. If  $\tilde{\gamma} = \sum_i m_j \tilde{\lambda}_j \in \tilde{\Gamma}$ , one has  $V(\tilde{\gamma}) = U(\tilde{\mathfrak{g}}).h^{\tilde{\gamma}}$  where  $h^{\tilde{\gamma}} = h_0^{m_0} \dots h_r^{m_r}$  is a highest weight vector of  $V(\tilde{\gamma})$ . In particular:  $h_j = h^{\tilde{\lambda}_j}$ ,  $d(\tilde{\gamma}) = \sum_j m_j d(\tilde{\lambda}_j)$ .

– The representation  $(\tilde{G} : V)$  is a prehomogeneous vector space. Let  $f_0, \dots, f_m$  be the basic relative invariants of this PHV and let  $\chi_j \in \mathbf{X}(\tilde{G}) = \mathbf{X}(C)$ ,  $0 \leq j \leq m$ , be their weights. After identification of  $\mathbf{X}(C)$  with a subgroup of  $\tilde{\Lambda}$  as above, one can number the  $\tilde{\lambda}_j$  so that

$$\tilde{\lambda}_0 \equiv \chi_0, \dots, \tilde{\lambda}_m \equiv \chi_m, \quad h_0 = f_0, \dots, h_m = f_m,$$

thus  $V(\tilde{\lambda}_j) = V(\chi_j)$  is the one dimensional  $\tilde{G}$ -module  $\mathbb{C}f_j$ .

Let  $p : \tilde{\Lambda}^+ = \Lambda^+ \oplus \mathbf{X}(C) \rightarrow \Lambda^+$  be the natural projection. Set

$$\tilde{\Gamma} = \Gamma_0 \bigoplus \Gamma, \quad \Gamma_0 = \bigoplus_{j=0}^m \mathbb{N}\tilde{\lambda}_j = \bigoplus_{j=0}^m \mathbb{N}\chi_j, \quad \Gamma = \bigoplus_{j=m+1}^r \mathbb{N}\tilde{\lambda}_j. \quad (4.1)$$

Using the results above, the next lemma is easy to prove.

**Lemma 4.2.** *One has:*

- $\Gamma_0 = \mathbf{X}(C) \cap \tilde{\Gamma} = \{\tilde{\gamma} \in \tilde{\Gamma} : \tilde{\gamma}(\mathbf{t}) = 0\}$ ;  $p$  induces a bijection  $\Gamma \xrightarrow{\sim} p(\Gamma)$ ;
- let  $\tilde{\gamma} \in \tilde{\Gamma}$ , then the  $G$ -module  $V(\tilde{\gamma})$  is isomorphic to  $E(p(\tilde{\gamma}))$ ;
- let  $\gamma, \gamma' \in \Gamma$ , then the following are equivalent:
  - $\gamma = \gamma'$
  - $p(\gamma) = p(\gamma')$
  - $V(\gamma) \cong V(\gamma')$  as  $G$ -modules;

(d) the algebra  $\mathbb{C}[V]^G$  of  $G$ -invariants is polynomial ring, more precisely:

$$\mathbb{C}[V]^G = \mathbb{C}[f_0, \dots, f_m] = \bigoplus_{\gamma \in \Gamma_0} \mathbb{C}h^\gamma.$$

Set:

$$H(V^*) = \bigoplus_{\gamma \in \Gamma} V(\gamma). \quad (4.2)$$

**Lemma 4.3.** *The multiplication map:*

$$\mathfrak{m} : H(V^*) \otimes_{\mathbb{C}} S(V^*)^G \longrightarrow S(V^*) = \mathbb{C}[V]$$

*is an isomorphism of  $G$ -modules.*

*Proof.* Let  $\tilde{\gamma} = \gamma + \gamma_0$ ,  $\gamma \in \Gamma$ ,  $\gamma_0 \in \Gamma_0$ . Observe that  $C$  acts by scalars on the simple  $\tilde{G}$ -module  $V(\tilde{\gamma})$ ; thus, since  $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}$ , we have:  $V(\tilde{\gamma}) = U(\tilde{\mathfrak{g}}).h^{\tilde{\gamma}} = U(\mathfrak{g})U(\mathfrak{c}).h^{\tilde{\gamma}} = U(\mathfrak{g}).h^{\tilde{\gamma}} = U(\mathfrak{g}).h^\gamma h^{\gamma_0} = (U(\tilde{\mathfrak{g}}).h^\gamma)h^{\gamma_0} = V(\gamma)h^{\gamma_0}$ . Therefore  $V(\tilde{\gamma}) = \mathfrak{m}(V(\gamma) \otimes \mathbb{C}h^{\gamma_0})$  with  $V(\gamma) \subset H(V^*)$ ,  $h^{\gamma_0} \in S(V^*)^G$ . Suppose that  $\mathfrak{m}(\sum_i v_i \otimes h^{\mu_i}) = 0$  with  $\mu_i \in \Gamma_0$ ,  $v_i \in V(\lambda_i)$ , the  $\lambda_i \in \Gamma$  being pairwise distinct. Observe that  $\lambda_i + \mu_i = \lambda_j + \mu_j$  forces  $\lambda_i - \lambda_j = \mu_j - \mu_i \in \Gamma \cap \Gamma_0 = (0)$ . Therefore  $v_i h^{\mu_i} \in V(\lambda_i + \mu_i)$  and  $\sum_i v_i h^{\mu_i} = 0$  yield  $v_i h^{\mu_i} = 0$ , hence  $v_i = 0$  for all  $i$ .  $\square$

Recall that we identify  $S(V)$  with the algebra of differential operators with constant coefficients. Consider the non-degenerate pairing

$$S(V) \otimes S(V^*) \longrightarrow \mathbb{C}, \quad u \otimes \varphi \mapsto \langle u \mid \varphi \rangle = u(\varphi)(0),$$

which extends the duality pairing  $V \otimes V^* \rightarrow \mathbb{C}$ . It is easily shown that:

- $\langle u \mid S^j(V^*) \rangle = 0$  if  $u \in S^i(V)$  and  $i \neq j$ ;
- $\langle \mid \rangle$  is  $\tilde{G}$ -equivariant.

Therefore  $u \mapsto \langle u \mid \rangle$  gives a  $\tilde{G}$ -isomorphism from  $S^i(V)$  onto  $S^i(V^*)^*$ . In particular, the representation  $(\tilde{G} : V^*)$  is MF and we can write:

$$S^i(V) = \bigoplus_{\{\tilde{\gamma} \in \tilde{\Gamma}, d(\tilde{\gamma})=i\}} Y(\tilde{\gamma}), \quad Y(\tilde{\gamma}) \cong V(\tilde{\gamma})^* \cong E(\tilde{\gamma}^*).$$

Hence,  $Y(\tilde{\gamma}) = U(\tilde{\mathfrak{g}}).\Delta^{\tilde{\gamma}}$  where  $\Delta^{\tilde{\gamma}}$  is a lowest weight vector (of weight  $-\tilde{\gamma}$ ). When  $\tilde{\gamma} = \tilde{\lambda}_j$  we set  $\Delta^{\tilde{\lambda}_j} = \Delta_j$ . Note that  $\Delta^{\tilde{\gamma}} = \prod_{i=0}^r \Delta_i^{m_i}$  if  $\tilde{\gamma} = \sum_i m_i \tilde{\lambda}_i$ .

If  $0 \leq i \leq m$  we have  $Y(\tilde{\lambda}_i) = \mathbb{C}\Delta_i$  where  $\Delta_i$  has weight  $-\lambda_i \equiv \chi_i^{-1}$ . Clearly, we may take  $\Delta_i = \partial(f_i^*)$  where  $f_i^*$  is the relative invariant constructed as in §3.1. We then have

$$S(V)^G = \mathbb{C}[\Delta_0, \dots, \Delta_m] = \bigoplus_{\gamma \in \Gamma_0} \mathbb{C}\Delta^\gamma \quad (4.3)$$

(which is a polynomial ring).

If  $\mu = \sum_i m_i \tilde{\lambda}_i$  and  $\nu = \sum_i n_i \tilde{\lambda}_i$  are elements of  $\tilde{\Gamma}$ , we say that  $\mu \leq \nu$  if  $m_i \leq n_i$  for all  $i$ . Let  $k : \tilde{\Gamma} \rightarrow \Gamma_0$  be the projection associated to the decomposition defined in (4.1); thus each  $\tilde{\lambda} \in \tilde{\Lambda}$  writes uniquely  $\gamma + k(\tilde{\lambda})$ ,  $\gamma \in \Gamma$ ,  $k(\tilde{\lambda}) \in \Gamma_0$ .

**Lemma 4.4.** *Let  $\tilde{\lambda} \in \Gamma_0$  and  $\tilde{\gamma} \in \tilde{\Gamma}$ . Then:*

- (a)  $\Delta^{\tilde{\gamma}}(h^{\tilde{\gamma}}) \neq 0$ ;
- (b)  $\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}}) \neq 0 \iff \tilde{\lambda} \leq k(\tilde{\gamma})$ , and in this case  $\Delta^{\tilde{\lambda}}$  gives an isomorphism of  $G$ -modules,  $\varphi \mapsto \Delta^{\tilde{\lambda}}(\varphi)$ , from  $V(\tilde{\gamma})$  onto  $V(\tilde{\gamma} - \tilde{\lambda})$ .

*Proof.* Set  $\tilde{\lambda} = \sum_{i=0}^m p_i \tilde{\lambda}_i$ ,  $\tilde{\gamma} = \sum_{i=0}^r q_i \tilde{\lambda}_i$ .

(a) Recall that we have an isomorphism of  $\tilde{G}$ -modules,  $\beta : Y(\tilde{\gamma}) \xrightarrow{\sim} V(\tilde{\gamma})^*$ ,  $\beta(u) = \langle u \mid \rangle$ . Thus  $\beta(\Delta^{\tilde{\gamma}})$  is a lowest vector in  $V(\tilde{\gamma})^*$ , which implies  $\beta(\Delta^{\tilde{\gamma}})(h^{\tilde{\gamma}}) =$

$\Delta^{\tilde{\gamma}}(h^{\tilde{\gamma}})(0) \neq 0$ . But  $\Delta^{\tilde{\gamma}} \in S^{d(\tilde{\gamma})}(V)$  where  $d(\tilde{\gamma})$  is the degree of  $\tilde{\gamma}$ , therefore  $\Delta^{\tilde{\gamma}}(h^{\tilde{\gamma}}) \in \mathbb{C}$ . Thus  $\Delta^{\tilde{\gamma}}(h^{\tilde{\gamma}}) = \Delta^{\tilde{\gamma}}(h^{\tilde{\gamma}})(0) \neq 0$ .

(b) Since  $\Delta^{\tilde{\lambda}} \in S(V)^G$  we have  $\Delta^{\tilde{\lambda}}(V(\tilde{\gamma})) = \Delta^{\tilde{\lambda}}(U(\mathfrak{g})h^{\tilde{\gamma}}) = U(\mathfrak{g})\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}})$ . By Lemma 4.2 we know that  $V(\tilde{\gamma})$  is a simple  $G$ -module, it follows that the map  $\Delta^{\tilde{\lambda}} : V(\tilde{\gamma}) \rightarrow \Delta^{\tilde{\lambda}}(V(\tilde{\gamma}))$  is either 0 or an isomorphism of  $G$ -modules.

Notice that  $\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}}) \in \mathbb{C}[V]^U$  has weight  $\tilde{\gamma} - \tilde{\lambda} = k(\tilde{\gamma}) - \tilde{\lambda} + \sum_{i=m+1}^r q_i \tilde{\lambda}_i$  where  $k(\tilde{\gamma}) - \tilde{\lambda} = \sum_{i=0}^m (q_i - p_i) \tilde{\lambda}_i$ . Therefore if  $q_i < p_i$  for some  $i = 0, \dots, m$  we must have  $\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}}) = 0$ , i.e.  $\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}}) \neq 0$  implies  $\tilde{\lambda} \leq k(\tilde{\gamma})$ . Conversely, suppose that  $\tilde{\lambda} \leq k(\tilde{\gamma})$ ; then, by (a),

$$0 \neq \Delta^{\tilde{\gamma}}(h^{\tilde{\gamma}}) = \Delta^{\tilde{\gamma}-\tilde{\lambda}}\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}})$$

which forces  $\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}}) \neq 0$ .

Now assume  $\tilde{\lambda} \leq k(\tilde{\gamma})$ . Then  $0 \neq \Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}}) \in \mathbb{C}[V]^U$  implies that  $\Delta^{\tilde{\lambda}}(h^{\tilde{\gamma}})$  is a highest weight vector in  $V(\tilde{\gamma} - \tilde{\lambda})$ , hence  $\Delta^{\tilde{\lambda}} : V(\tilde{\gamma}) \xrightarrow{\sim} \Delta^{\tilde{\lambda}}(V(\tilde{\gamma})) = V(\tilde{\gamma} - \tilde{\lambda})$ .  $\square$

Recall the definition of  $H(V^*)$  given in (4.2) and set  $S_+(V) = \bigoplus_{i>0} S^i(V)$ . The next proposition identifies  $H(V^*)$  with harmonic elements.

**Proposition 4.5.** *We have:*

$$\begin{aligned} H(V^*) &= \{\varphi \in \mathbb{C}[V] : \Delta_0(\varphi) = \dots = \Delta_m(\varphi) = 0\} \\ &= \{\varphi \in \mathbb{C}[V] : D(\varphi) = 0 \text{ for all } D \in S_+(V)^G\}. \end{aligned}$$

*Proof.* From (4.3) we know that  $S_+(V)^G = \bigoplus_{0 \neq \tilde{\lambda} \in \Gamma_0} \mathbb{C}\Delta^{\tilde{\lambda}}$ . Let  $\varphi \in V(\tilde{\gamma})$  for some  $\tilde{\gamma} \in \Gamma$  and let  $0 \neq \tilde{\lambda} \in \Gamma_0$ . We have  $k(\tilde{\gamma}) = 0$ , thus  $\Delta^{\tilde{\lambda}}(\varphi) = 0$  by Lemma 4.4(b). This shows that  $H(V^*) \subset \{\varphi \in \mathbb{C}[V] : D(\varphi) = 0 \text{ for all } D \in S_+(V)^G\}$ .

Conversely assume that  $\varphi = \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \varphi_{\tilde{\gamma}}$ ,  $\varphi_{\tilde{\gamma}} \in V(\tilde{\gamma})$ , satisfies  $\Delta_i(\varphi) = 0$  for all  $i = 0, \dots, m$ . Fix  $i \in \{0, \dots, m\}$ . By Lemma 4.4(b) we get that  $\Delta_i(V(\tilde{\gamma})) = 0$  if  $\tilde{\lambda}_i \not\leq k(\tilde{\gamma})$  and  $\Delta_i : V(\tilde{\gamma}) \xrightarrow{\sim} V(\tilde{\gamma} - \tilde{\lambda}_i)$  if  $\tilde{\lambda}_i \leq k(\tilde{\gamma})$ . Therefore  $\Delta_i(\varphi) = \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \Delta_i(\varphi_{\tilde{\gamma}})$  belongs to  $\bigoplus_{\{\tilde{\gamma} \in \tilde{\Gamma}, \tilde{\lambda}_i \leq k(\tilde{\gamma})\}} V(\tilde{\gamma} - \tilde{\lambda}_i)$ . Since  $\Delta_i(\varphi) = 0$  we can deduce that  $\Delta_i(\varphi_{\tilde{\gamma}}) = 0$  for all  $\tilde{\gamma}$  such that  $\tilde{\lambda}_i \leq k(\tilde{\gamma})$ . By the previous remark this implies  $\varphi_{\tilde{\gamma}} = 0$  when  $\tilde{\lambda}_i \leq k(\tilde{\gamma})$ , thus  $\varphi = \sum_{\{\tilde{\gamma} \in \tilde{\Gamma}, \tilde{\lambda}_i \not\leq k(\tilde{\gamma})\}} \varphi_{\tilde{\gamma}}$ . Observe that  $\tilde{\lambda}_i \not\leq k(\tilde{\gamma})$  means that the weight  $\tilde{\lambda}_i$  does not appear in  $\tilde{\gamma}$ . Since this holds for all  $i = 0, \dots, m$  we deduce that  $\varphi = \sum_{\tilde{\gamma} \in \Gamma} \varphi_{\tilde{\gamma}}$ . Hence the result.  $\square$

**Remark 4.6.** If we set  $H(V) = \bigoplus_{\gamma \in \Gamma} Y(\gamma)$  we obtain that  $S(V) \cong H(V) \otimes S(V)^G$  as  $G$ -modules, with an analogous characterization of  $H(V)$ .

We now recall some facts about invariant differential operators on MF representations, cf. [1, 18, 26]. Recall that the  $\mathbb{C}[V]$ -module  $\mathcal{D}(V)$  identifies with  $S(V^*) \otimes S(V)$  through the multiplication map

$$\mathfrak{m} : S(V^*) \otimes S(V) \xrightarrow{\sim} \mathcal{D}(V), \quad \varphi \otimes f \mapsto \varphi f(\partial).$$

The isomorphism  $\mathfrak{m}$  is also  $\tilde{G}$ -equivariant, hence  $\mathcal{D}(V)^{\tilde{G}} \cong \bigoplus_{\tilde{\gamma} \in \tilde{\Gamma}} [V(\tilde{\gamma}) \otimes Y(\tilde{\gamma})]^{\tilde{G}}$ . But, since  $Y(\tilde{\gamma}) \cong V(\tilde{\gamma})^*$ ,  $[V(\tilde{\gamma}) \otimes Y(\tilde{\gamma})]^{\tilde{G}} = \mathbb{C}E_{\tilde{\gamma}}$  is one dimensional. Let

$$E_{\tilde{\gamma}}(x, \partial_x) = \frac{1}{\dim V(\tilde{\gamma})} \mathfrak{m}(E_{\tilde{\gamma}}) \in \mathcal{D}(V)^{\tilde{G}}$$

be the operator corresponding to  $E_{\tilde{\gamma}}$ . The  $E_{\tilde{\gamma}}(x, \partial_x)$  are called the *normalized Capelli operators*. Set

$$E_j = E_{\tilde{\lambda}_j}(x, \partial_x), \quad 0 \leq j \leq r. \quad (4.4)$$

It is known [18, Proposition 7.1] that  $(\tilde{G} : V)$  multiplicity free is equivalent to  $\mathcal{D}(V)^{\tilde{G}}$  commutative. The operators  $E_j$  give a set of generators for this algebra, cf. [18, Theorem 9.1] or [1, Corollary 7.4.4]:

**Theorem 4.7** (Howe-Umeda).  $\tilde{\mathbb{D}} = \mathcal{D}(V)^{\tilde{G}} = \mathbb{C}[E_0, \dots, E_r] = \bigoplus_{\tilde{\gamma} \in \tilde{\Gamma}} \mathbb{C}E_{\tilde{\gamma}}(x, \partial_x)$  is a commutative polynomial ring.

Notice for further use the following property of the Capelli operators, see [26, Corollary 4.4] or [1, Proposition 8.3.2]:

**Proposition 4.8.** Set  $\mathfrak{a}^* = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}\tilde{\Gamma} = \bigoplus_{i=0}^r \mathbb{C}\tilde{\lambda}_i$ ,  $\mathfrak{a} = \bigoplus_{i=0}^r \mathbb{C}a_i$  where  $\{a_i\}_i$  is the dual basis of  $\{\tilde{\lambda}_i\}_i$ . For each  $\tilde{\gamma} \in \tilde{\Gamma}$  there exists a polynomial function  $b_{\tilde{\gamma}} = b_{\tilde{\gamma}}(a_0, \dots, a_r) \in \mathbb{C}[\mathfrak{a}^*] = S(\mathfrak{a}) = \mathbb{C}[a_0, \dots, a_r]$  such that

$$E_{\tilde{\gamma}}(x, \partial_x)(h^{\tilde{\lambda}}) = b_{\tilde{\gamma}}(\tilde{\lambda})h^{\tilde{\lambda}} \text{ for all } \tilde{\lambda} \in \mathfrak{a}^*.$$

**Remarks 4.9.** (1) Suppose that  $(\tilde{G} : V)$  is irreducible. Then we can assume that  $V = V(\tilde{\gamma}_r)$ . If  $\dim V = N$  we have  $E_r = E_r(x, \partial_x) = \tilde{\Theta} = \frac{1}{N}\Theta$  where  $\Theta$  is the Euler vector field.

(2) If  $j \in \{0, \dots, m\}$  we may take  $E_j = f_j \Delta_j$ . Recall that  $f_j = h^{\tilde{\lambda}_j}$  and  $\Delta_j = \partial(f_j^*)$ . By Theorem 3.1 there exists  $b_j(s) \in \mathbb{C}[s]$  such that  $\Delta_j(f_j^m) = b_j^*(m)f_j^{m-1}$ ; thus  $E_j(f_j^m) = b_j^*(m)f_j^m$ . This shows that  $b_{\tilde{\lambda}_j}(s, 0, \dots, 0) = b_j^*(s)$ .

(3) Let  $D \in \tilde{\mathbb{D}}$ ; then  $D(V(\tilde{\lambda})) = U(\tilde{\mathfrak{g}}).D(h^{\tilde{\lambda}})$  is either (0) or equal to  $V(\tilde{\lambda})$ . Indeed, the  $\tilde{G}$ -invariance of  $D$  implies that  $g.D(h^{\tilde{\lambda}}) = D(g.h^{\tilde{\lambda}}) = \tilde{\lambda}(g)D(h^{\tilde{\lambda}})$  for all  $g \in \tilde{T}U$ , where we have considered here  $\tilde{\lambda}$  as a character of the Borel subgroup  $\tilde{T}U$  of  $\tilde{G}$ ; thus  $D(h^{\tilde{\lambda}}) \in \mathbb{C}h^{\tilde{\lambda}}$  is either 0 or a highest weight vector of  $V(\tilde{\lambda})$ .

**4.2. MF representations with a one dimensional quotient.** In this subsection we will work under the following hypothesis:

**Hypothesis B.**  $(\tilde{G} : V)$  is a multiplicity free representation which satisfies Hypothesis A.

In the notation of §4.1, this condition means that  $m = 0$ , i.e.  $\Gamma_0 = \mathbb{N}\tilde{\lambda}_0$ . Set  $f = f_0$ ,  $n = d(\tilde{\lambda}_0)$ ,  $\Delta = \Delta_0 = \partial(f^*)$ , then we have  $f \in S^n(V^*)$ ,  $V(\tilde{\lambda}_0) = \mathbb{C}f$ ,  $Y(\tilde{\lambda}_0) = \mathbb{C}\Delta$  and:

$$\mathbb{C}[V]^G = \mathbb{C}[f], \quad S(V)^G = \mathbb{C}[\Delta]$$

(see Lemma 4.2 and (4.3)). By Remark 4.9(2) we have  $E_0 = f\Delta$ ,  $b_{\tilde{\lambda}_j}(s, 0, \dots, 0) = b^*(s) = b(s-1)$  where  $b(s)$  is the  $b$ -function of  $f$ . Recall from (3.8) the following notation:

$$\mathbb{D} = \mathcal{D}(V)^G \supset \mathcal{D}(V)^{\tilde{G}} = \tilde{\mathbb{D}}.$$

Recall also that  $(G : V)$  is polar and that we have studied in §3.2 the image of radial component map  $\text{rad} : \mathcal{D}(V)^G \rightarrow \mathcal{D}(\mathfrak{h}/W) = \mathbb{C}[z, \partial_z]$ . We now want to describe  $J = \text{Ker}(\text{rad})$ .

**Lemma 4.10.** Let  $P \in \tilde{\mathbb{D}}$ . Then there exists a polynomial  $b_P(s) \in \mathbb{C}[s]$  such that

$$P(f^m) = b_P(m)f^m, \quad \text{rad}(P) = b_P(\theta), \quad P - b_P(\bar{\Theta}) \in J = \text{Ker}(\text{rad}).$$

*Proof.* Write  $P = \sum_{\tilde{\gamma} \in \tilde{\Gamma}} p_{\tilde{\gamma}} E_{\tilde{\gamma}}(x, \partial_x)$ , cf. Theorem 4.7, and define a polynomial function by  $b_P(s) = \sum_{\tilde{\gamma} \in \tilde{\Gamma}} p_{\tilde{\gamma}} b_{\tilde{\gamma}}(s, 0, \dots, 0)$ , where  $b_{\tilde{\gamma}} \in S(\mathfrak{a})$  is as in Proposition 4.8. Since  $f^m = h^{m\tilde{\lambda}_0}$  we obtain that  $P(f^m) = b_P(m)f^m$ . It follows that  $\text{rad}(P)(z^m) = b_P(m)z^m$  for all  $m \in \mathbb{N}$  and Lemma 2.1 yields  $\text{rad}(P) = b_P(\theta)$ . Since  $\text{rad}(\bar{\Theta}) = \theta$  we have  $\text{rad}(P - b_P(\bar{\Theta})) = 0$ .  $\square$

Notice that  $\bar{\Theta} \in \tilde{\mathbb{D}}$ ; for  $j \in \{0, \dots, r\}$  we set

$$\Omega_j = E_j - b_{E_j}(\bar{\Theta}) \in J \cap \tilde{\mathbb{D}}. \quad (4.5)$$

Thus we have:

$$\tilde{\mathbb{D}} = \mathbb{C}[E_0, \dots, E_r] = \mathbb{C}[\Omega_0, \Omega_1, \dots, \Omega_r, \bar{\Theta}].$$

Recall that for  $j = 0$  one has  $E_0 = f\Delta$ , hence  $b_{E_0}(s) = b^*(s)$  where  $b(s)$  is the  $b$ -function of  $f$ . Thus  $\Omega_0 = f\Delta - b^*(\bar{\Theta})$ ; observe that we have already shown in §3.2 that  $\text{rad}(f\Delta - b^*(\bar{\Theta})) = z\delta - b^*(\theta) = 0$ .

When  $V$  is irreducible we adopt the notation of Remark 4.9(1) and we obtain  $E_r = \bar{\Theta}$ ,  $b_{E_r}(s) = s$ , thus  $\Omega_r = 0$ . Therefore in this case one has

$$\tilde{\mathbb{D}} = \mathbb{C}[\bar{\Theta}, \Omega_0, \dots, \Omega_{r-1}]. \quad (4.6)$$

The next result gives a description of  $\text{Ker}(\text{rad})$  and another proof of Theorem 3.9 in the case of MF representations. When  $(\tilde{G} : V) = (\text{GL}(n) : S^2\mathbb{C}^n)$ , part (i) of Theorem 4.11 is proved in [36, Proposition 2.1].

**Theorem 4.11.** *The following properties hold.*

- (i)  $\mathbb{D} = \tilde{\mathbb{D}}[f, \Delta] = \mathbb{C}[E_1, \dots, E_r][f, \Delta] = \mathbb{C}[\Omega_1, \dots, \Omega_r][f, \Delta, \bar{\Theta}]$ .
- (ii)  $\mathbb{D} = (\bigoplus_{p \in \mathbb{N}} \tilde{\mathbb{D}} f^p) \oplus (\bigoplus_{p \in \mathbb{N}^*} \tilde{\mathbb{D}} \Delta^p)$ .
- (iii) For  $k \in \mathbb{Z}$ , set

$$\mathbb{D}[k] = \begin{cases} \tilde{\mathbb{D}} f^k & \text{if } k \geq 0, \\ \tilde{\mathbb{D}} \Delta^{-k} & \text{if } k < 0; \end{cases}$$

then  $\mathbb{D}[k] = f^k \tilde{\mathbb{D}}$ , if  $k \geq 0$ , or  $\Delta^{-k} \tilde{\mathbb{D}}$ , if  $k < 0$ .

- (iv)  $R = \text{rad}(\mathbb{D}) = U = \mathbb{C}[z, \delta, \theta]$ .

- (v)  $J = \text{Ker}(\text{rad}) = \sum_{i=0}^r \mathbb{D} \Omega_i = \sum_{i=0}^r \Omega_i \mathbb{D}$ .

*Proof.* Endow  $\mathcal{D}(V)$ ,  $\mathbb{D}$  and  $\tilde{\mathbb{D}}$  with the “Bernstein filtration”, i.e.:

$$\mathcal{F}_p \mathcal{D}(V) = \sum_{i+j \leq p} S^i(V^*) S^j(V), \quad \mathcal{F}_p \mathbb{D} = \mathcal{F}_p \mathcal{D}(V) \cap \mathbb{D} \supset \mathcal{F}_p \tilde{\mathbb{D}} = \mathcal{F}_p \mathcal{D}(V) \cap \tilde{\mathbb{D}}.$$

Then, since  $\tilde{G}$  and  $G$  are reductive,

$$\tilde{\mathbb{S}} = [\text{gr}_{\mathcal{F}} \mathcal{D}(V)]^{\tilde{G}} = [S(V^*) \otimes S(V)]^{\tilde{G}}, \quad \mathbb{S} = [\text{gr}_{\mathcal{F}} \mathcal{D}(V)]^G = [S(V^*) \otimes S(V)]^G.$$

Denote by  $\sigma_j = \text{gr}_{\mathcal{F}}(E_j) \in [V(\tilde{\lambda}_j) \otimes Y(\tilde{\lambda}_j)]^{\tilde{G}}$  the principal symbol of  $E_j$  for  $\mathcal{F}$ . Then  $\tilde{\mathbb{S}} = \mathbb{C}[\sigma_0, \dots, \sigma_r]$ , see for example [1]. Recall that  $E_0 = f\Delta$ , hence  $\sigma_0 = f f^*$ . By Lemma 4.3 and Remark 4.6 we know that  $S(V^*) = H(V^*) \otimes \mathbb{C}[f]$ ,  $S(V) = H(V) \otimes \mathbb{C}[f^*]$ , hence  $\mathbb{S} = [H(V^*) \otimes H(V)]^G \otimes \mathbb{C}[f, f^*]$  (as vector spaces). Let  $\gamma, \lambda \in \Gamma$ ; recall that the  $G$ -module  $V(\gamma)$  is isomorphic to  $E(p(\gamma))$  and that  $p(\gamma) = p(\lambda)$  if and only if  $\gamma = \lambda$ , cf. Lemma 4.2. It follows that  $[V(\gamma) \otimes Y(\gamma)]^G = [V(\gamma) \otimes Y(\gamma)]^{\tilde{G}} = \mathbb{C} E_\gamma$  and

$$[H(V^*) \otimes H(V)]^G = \bigoplus_{\gamma \in \Gamma} \mathbb{C} E_\gamma \subset \tilde{\mathbb{S}} = \mathbb{C}[\sigma_0, \dots, \sigma_r].$$

Thus:

$$\tilde{\mathbb{S}}[f, f^*] \subset \mathbb{S} = [H(V^*) \otimes H(V)]^G \otimes \mathbb{C}[f, f^*] \subset \tilde{\mathbb{S}}[f, f^*].$$

Since the centre  $C$  of  $\tilde{G}$  acts trivially on  $\tilde{\mathbb{S}}$  and via  $\chi^j$ , resp.  $\chi^{-j}$ , on  $f^j$ , resp.  $(f^*)^j$ , we obtain:

$$\mathbb{S} = \tilde{\mathbb{S}}[f, f^*] = \left( \bigoplus_{j \geq 0} \tilde{\mathbb{S}} f^j \right) \oplus \left( \bigoplus_{i > 0} \tilde{\mathbb{S}} (f^*)^i \right) = \mathbb{C}[\sigma_1, \dots, \sigma_r] \otimes \mathbb{C}[f, f^*]. \quad (*)$$

Then, by a filtration argument, one deduces that  $\mathbb{D} = \sum_p \tilde{\mathbb{D}} f^p + \sum_p \tilde{\mathbb{D}} \Delta^p = \tilde{\mathbb{D}}[f, \Delta] = \mathbb{C}[E_1, \dots, E_r][f, \Delta] = \mathbb{C}[\Omega_1, \dots, \Omega_r][f, \Delta, \bar{\Theta}]$  (recall that  $\Omega_j = E_j - b_{E_j}(\bar{\Theta})$ ). This proves (i).

Observe that  $\tilde{\mathbb{D}} f^p$  and  $f^p \tilde{\mathbb{D}}$ , resp.  $\tilde{\mathbb{D}} \Delta^p$  and  $\Delta^p \tilde{\mathbb{D}}$ , are contained in the  $\chi^p$ -weight space, resp.  $\chi^{-p}$ -weight space, for the action of  $C$  on  $\mathbb{D}$ . This implies easily, as in  $(\star)$ , that these one dimensional subspaces are equal to the corresponding weight spaces. This proves (ii) and (iii).

Since  $\Omega_i \in J = \text{Ker}(\text{rad})$ , we obtain  $\text{rad}(\mathbb{D}) = \mathbb{C}[\text{rad}(f), \text{rad}(\Delta), \text{rad}(\bar{\Theta})] = \mathbb{C}[z, \delta, \theta]$ , hence (iv).

(v) Let  $P \in \mathbb{D}$  and write  $P = \sum_k P_k$ ,  $P_k \in \mathbb{D}[k]$  with  $P_k = Q_k f^k$  or  $Q'_k \Delta^{-k}$  and  $Q_k, Q'_k \in \tilde{\mathbb{D}} = \mathbb{C}[\Omega_0, \dots, \Omega_r, \bar{\Theta}]$ . Set:

$$Q_p = \sum_{i \geq 0} Q_{p,i} \bar{\Theta}^i, \quad Q_p = \sum_{i \geq 0} Q'_{p,i} \bar{\Theta}^i$$

where  $Q_{p,i}, Q'_{p,i} \in \mathbb{C}[\Omega_0, \dots, \Omega_r]$ . Since  $Q_{p,i} \in Q_{p,i}(0) + \sum_j \Omega_j \tilde{\mathbb{D}}$ ,  $Q'_{p,i} \in Q'_{p,i}(0) + \sum_j \Omega_j \tilde{\mathbb{D}}$ ,  $Q_{p,i}(0), Q'_{p,i}(0) \in \mathbb{C}$ , we obtain by applying  $\text{rad}$ :

$$\text{rad}(P) = \sum_{k \geq 0} \left( \sum_{i \geq 0} Q_{k,i}(0) \theta^i \right) z^k + \sum_{p > 0} \left( \sum_{i \geq 0} Q'_{p,i}(0) \theta^i \right) \delta^p.$$

Recall from §2.2 that there exists a filtration on  $R$  such that  $\text{gr}(R)$  is isomorphic to  $\mathbb{C}[X, Y, S]/(XY - S^n)$  where  $\text{gr}(z) \equiv X, \text{gr}(\delta) \equiv Y, \text{gr}(\theta) \equiv S$  (up to some scalar). This implies easily that  $R = (\bigoplus_{k \geq 0} \mathbb{C}[\theta] z^k) \oplus (\bigoplus_{p > 0} \mathbb{C}[\theta] \delta^p)$ . Now suppose that  $P \in J$ , then  $\text{rad}(P) = 0 = \sum_{k \geq 0} (\sum_{i \geq 0} Q_{k,i}(0) \theta^i) z^k + \sum_{p > 0} (\sum_{i \geq 0} Q'_{p,i}(0) \theta^i) \delta^p$  forces  $\sum_{i \geq 0} Q_{k,i}(0) \theta^i = \sum_{i \geq 0} Q'_{p,i}(0) \theta^i = 0$ , hence  $Q_{k,i}(0) = Q'_{p,i}(0) = 0$  for all  $k, p, i$ . This shows that  $Q_k, Q'_p \in \sum_{i=0}^r \Omega_i \tilde{\mathbb{D}} + \dots + \Omega_r \tilde{\mathbb{D}}$  and therefore  $P \in \sum_{i=0}^r \Omega_i \tilde{\mathbb{D}}$ . Writing  $P_k \in f^k \tilde{\mathbb{D}}$  or  $\Delta^{-k} \tilde{\mathbb{D}}$  yields  $P \in \sum_{i=0}^r \tilde{\mathbb{D}} \Omega_i + \dots + \tilde{\mathbb{D}} \Omega_r$ .  $\square$

**Remark 4.12.** (1) In the case where  $(\tilde{G} : V)$  is irreducible we have noticed that  $\Omega_r = 0$ , see (4.6), thus

$$J = \sum_{i=0}^{r-1} \mathbb{D} \Omega_i = \sum_{i=0}^{r-1} \Omega_i \mathbb{D}.$$

(2) From  $[\bar{\Theta}, f^k] = k f^k$  and  $[\bar{\Theta}, \Delta^k] = -k \Delta^k$  we can deduce that

$$\mathbb{D}[k] = \{D \in \mathbb{D} : [\bar{\Theta}, D] = kD\}.$$

**4.3. A Howe duality.** The Howe duality for the Weil representation gives a bijection between irreducible finite dimensional representations of  $\text{SO}(n)$  and irreducible lowest weight  $U(\mathfrak{sl}(2))$ -modules. Algebraically, this result corresponds to the case  $(\tilde{G} = \text{SO}(n) \times \mathbb{C}^* : V = \mathbb{C}^n)$ : here the Lie subalgebra of  $\mathcal{D}(V)$  generated by  $f$  (quadratic form) and  $\Delta$  (Laplacian) is isomorphic to  $\mathfrak{sl}(2)$ . More precisely we have the following result. Let  $\mathcal{A} \cong \mathfrak{sl}(2)$  be this Lie algebra, denote by  $H_d \subset H(V^*)$  the space of harmonic polynomials of degree  $d$ , i.e.:

$$H_d = \{q \in S^d(V^*) : \Delta(q) = 0\}.$$

Each  $H_d$  is an irreducible  $\text{SO}(n)$ -module and the  $\mathcal{A} \times \text{SO}(n)$ -module  $\mathbb{C}[V]$  decomposes as

$$\mathbb{C}[V] = \bigoplus_d X(d + n/2) \otimes H_d$$

where  $X(d + n/2)$  is the irreducible lowest  $\mathfrak{sl}(2)$ -module of lowest weight  $d + n/2$ .

This kind of duality has been extended by H. Rubenthaler [49] to a class of PHV, the so-called *commutative parabolic* PHV. They are associated to short gradings  $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1$  on simple Lie algebras. The commutative parabolic PHV are

irreducible, MF and satisfy  $\dim V//G = 1$ , thus Hypothesis B holds. We want to generalize the Howe duality to the more general class of representations  $(\tilde{G} : V)$  satisfying only Hypothesis B. We therefore fix a representation  $(\tilde{G} : V)$  satisfying this hypothesis, see §4.2. We have indicated in the last column of the table of Appendix A the irreducible MF representations  $(\tilde{G} : V)$  which are of commutative parabolic type.

Let

$$\mathcal{A} = \text{Lie}\langle f, \Delta \rangle \subset (\mathcal{D}(V), [\ , \ ] ) \quad (4.7)$$

be the Lie subalgebra generated by  $f, \Delta$ . Notice that  $\mathcal{A} \subset \mathbb{D}$ . Let  $\tilde{\gamma} = \sum_{j=0}^r a_j \tilde{\lambda}_j$ . Recall that  $V(\tilde{\gamma}) = U(\mathfrak{g}).h^{\tilde{\gamma}}$ ; we then set:

$$\mathbf{a} = (a_0, \dots, a_r), \quad h^{\tilde{\gamma}} = h^{\mathbf{a}} = f^{a_0} h_1^{a_1} \dots h_r^{a_r}, \quad V(\tilde{\gamma}) = V(\mathbf{a}) = V(a_0, \dots, a_r).$$

Let  $b \in \mathbb{N}$ , by Lemma 4.4(b), the operator  $\Delta^b$  acts as follows:  $\Delta^b(V(\mathbf{a})) = 0$  if  $a_0 < b$ ,  $\Delta^b : V(a_0, \dots, a_r) \xrightarrow{\sim} V(a_0 - b, a_1, \dots, a_r)$  if  $b \leq a_0$ , and in the latter case  $\Delta^b(h^{\mathbf{a}}) \in \mathbb{C}^* f^{a_0-b} h_1^{a_1} \dots h_r^{a_r}$  is a highest weight vector in  $V(a_0 - b, \dots, a_r)$ . Obviously, the multiplication by  $f^b$  gives an isomorphism  $f^b : V(a_0, \dots, a_r) \xrightarrow{\sim} V(a_0 + b, a_1, \dots, a_r)$  of  $\tilde{G}$ -modules. Define:

$$\mathcal{A}_j = \{D \in \mathcal{A} : D(V(a_0, \dots, a_r)) \subset V(a_0 + j, a_1, \dots, a_r) \text{ for all } \mathbf{a} \in \mathbb{N}^{r+1}\}. \quad (4.8)$$

It is clear that  $\bigoplus_{j \in \mathbb{Z}} \mathcal{A}_j \subset \mathcal{A}$  and, by the previous remarks,  $f^b \in \mathcal{A}_b$ ,  $\Delta^b \in \mathcal{A}_{-b}$ .

*Remarks.* 1) It is difficult to compute in the Lie algebra  $\mathcal{A}$  because  $[\Delta, f] = \psi(-\bar{\Theta}) + Q$  for some  $Q \in \text{Ker}(\text{rad})$  which is not easy to calculate (recall that here  $\psi(s) = b(-s) - b(-s-1)$ , cf. (2.6)). For example when  $(\tilde{G} : V) = (\text{GL}(n) \times \text{SL}(n) : \text{M}_n(\mathbb{C}))$  P. Nang [41] has shown that (up to some scalar):  $Q = \text{trace}(\mathbf{x}^\# \bar{\partial}^\#)$  where  $\mathbf{x}^\#$ , resp.  $\bar{\partial}^\#$ , is the adjoint matrix of  $\mathbf{x} = [x_{ij}]_{ij}$ , resp.  $\bar{\partial} = [\partial_{x_{ij}}]_{ij}$ . (See also [43, Proposition 7] for the case  $(\text{GL}(2m) : \bigwedge^2 \mathbb{C}^{2m})$ .)  
2) When  $n = \deg f = 2$  one has  $\mathcal{A} \cong \mathfrak{sl}(2)$ , thus  $\dim_{\mathbb{C}} \mathcal{A} = 3$ .

The assertion (2) of the next proposition should be compared with [49, Théorème 3.1].

**Proposition 4.13.** (1) One has  $\mathcal{A}_k = \mathcal{A} \cap \mathbb{D}[k]$ ,  $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k$ . The Lie subalgebra  $\mathcal{A}_0$  is abelian.

(2) Suppose  $n \geq 3$ , then  $\dim_{\mathbb{C}} \mathcal{A}_k = \infty$  for all  $k \in \mathbb{Z}$ .

*Proof.* (1) Since  $f, \Delta \in \mathcal{A}$  the relations  $[\bar{\Theta}, f] = f$  and  $[\bar{\Theta}, \Delta] = -\Delta$  show that  $\text{ad}(\bar{\Theta}) : \mathbb{D} \rightarrow \mathbb{D}$  induces an endomorphism of  $\mathcal{A}$ . From  $\mathcal{A} \subset \mathbb{D}$  we deduce that  $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A} \cap \mathbb{D}[k]$ . let  $P \in \mathbb{D}[k]$ ,  $P = f^k D$  or  $\Delta^{-k} D$  be an element of  $\mathcal{A} \cap \mathbb{D}[k]$ . Then, using Remark 4.9(3), we see that  $P \in \mathcal{A}_k$ . The desired equalities follow easily. Since  $\mathcal{A}_0 \subset \tilde{\mathbb{D}}$  and  $\tilde{\mathbb{D}}$  is a commutative algebra, cf. Theorem 4.7,  $\mathcal{A}_0$  is abelian.

(2) We claim that  $\text{rad}(\mathcal{A}_j) = \mathcal{L}_j$ , where  $\mathcal{L}_j$  is defined as in §2.3, i.e.:  $\mathcal{L} = \text{Lie}\langle f, \Delta \rangle$ ,  $\mathcal{L}_i = \{u \in \mathcal{L} : u(z^m) \in \mathbb{C}z^{m+j} \text{ for all } m \in \mathbb{N}\}$  (with the convention that  $\mathbb{C}z^{m+j} = (0)$  when  $m+j < 0$ ). Note first that  $\text{rad}(\mathcal{A}) = \text{Lie}\langle \text{rad}(f), \text{rad}(\Delta) \rangle = \mathcal{L}$ . Let  $D \in \mathcal{A}_j$ , then  $\text{rad}(D)(z^m) = \psi(D(f^m))$ . Observe that  $f^m \in V(m, 0, \dots, 0)$ , hence  $D(f^m) \in V(m+j, 0, \dots, 0)$ , which is  $(0)$  if  $m+j < 0$  and  $\mathbb{C}f^{m+j}$  if  $m+j \geq 0$ . Thus  $\text{rad}(D) \in \mathbb{C}z^{m+j}$  and  $\text{rad}(\mathcal{A}_j) \subset \mathcal{L}_j$ . It follows that  $\text{rad}(\mathcal{A}) = \sum_j \text{rad}(\mathcal{A}_j) \subset \bigoplus_j \mathcal{L}_j \subset \mathcal{L} = \text{rad}(\mathcal{A})$ . Therefore  $\text{rad}(\mathcal{A}_j) = \mathcal{L}_j$  for all  $j$  (and  $\mathcal{L} = \bigoplus_j \mathcal{L}_j$ ). Now, Proposition 2.5 yields the desired assertion.  $\square$

Set  $\mathcal{A}_+ = \bigoplus_{k \geq 0} \mathcal{A}_k$ ,  $\mathcal{A}_- = \bigoplus_{k < 0} \mathcal{A}_k$ . We then have a triangular decomposition  $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+$  which enables us to introduce the notion of a lowest weight  $\mathcal{A}$ -module, see [49]. As usual in this situation the weights will be elements of the linear dual space  $\mathcal{A}_0^*$  of the abelian Lie algebra  $\mathcal{A}_0$ .

**Definition 4.14.** The  $\mathcal{A}$ -module  $X$  is a lowest weight module if there exist  $x \in X$  and  $\varphi \in \mathcal{A}_0^*$  such that:  $X = U(\mathcal{A}).x$ ,  $\mathcal{A}_-.x = 0$ ,  $a.x = \varphi(a)x$  for all  $a \in \mathcal{A}_0$ .

The next theorem generalizes [49, Proposition 4.2]; it gives a Howe duality for MF representations with a one dimensional quotient (see also [10, Corollary 4.5.17]). Recall that  $H(V^*) = \bigoplus_{\gamma \in \Gamma} V(\gamma)$ , where  $V(\gamma) \cong E(p(\gamma))$ , is equal to the space of harmonic elements, i.e.  $\{\varphi \in \mathbb{C}[V] : \Delta_0(\varphi) = \dots = \Delta_m(\varphi) = 0\}$ , cf. (4.2) and Proposition 4.5.

**Theorem 4.15.** Let  $(\tilde{G} : V)$  satisfying Hypothesis B. The  $\mathcal{A} \times \mathfrak{g}$ -module  $\mathbb{C}[V]$  decomposes as  $\mathbb{C}[V] \cong \bigoplus_{\gamma \in \Gamma} X(\gamma) \otimes E(p(\gamma))$ , where  $X(\gamma) = U(\mathcal{A}).h^\gamma$  is an irreducible lowest weight  $\mathcal{A}$ -module. Moreover:  $X(\gamma) \cong X(\gamma') \iff \gamma = \gamma'$ .

*Proof.* Recall that  $\tilde{\Gamma} = \mathbb{N}\tilde{\lambda}_0 \oplus \Gamma$ ,  $\Gamma = \bigoplus_{i=1}^r \mathbb{N}\tilde{\lambda}_i$ . Let  $\gamma = \sum_{i=1}^r a_i \tilde{\lambda}_i \in \Gamma$ ; set  $P_\gamma = \gamma + \mathbb{N}\tilde{\lambda}_0$ . Then  $h^\gamma = h_1^{a_1} \dots h_r^{a_r}$  and  $V(\gamma) = U(\mathfrak{g}).h^\gamma \cong E(p(\gamma))$ , see Lemma 4.2. From  $f^b h^\gamma = h^{(b, a_1, \dots, a_r)}$  and  $\Delta^b(f^{a_0} h_1^{a_1} \dots h_r^{a_r}) \in \mathbb{C}^* f^{a_0-b} h_1^{a_1} \dots h_r^{a_r}$  when  $a_0 \geq b$ , and 0 when  $a_0 < b$ , we get that  $X(\gamma) = U(\mathcal{A}).h^\gamma = \bigoplus_{\mu \in P_\gamma} \mathbb{C}h^\mu$  is an irreducible  $U(\mathcal{A})$ -module. Let  $D \in \mathcal{A}_{-j} = \mathcal{A} \cap \mathbb{D}[-j]$ ,  $j > 0$ , and write  $D = D' \Delta^j$ ,  $D' \in \tilde{\mathbb{D}}$ . Then  $D(h^\gamma) = D' \Delta^j(h^\gamma)$  and we have noticed that  $\Delta^j(h^\gamma) = 0$ , thus  $\mathcal{A}_-.h^\gamma = 0$ . When  $D \in \mathcal{A}_0 = \mathcal{A} \cap \tilde{\mathbb{D}}$ , Remark 4.9(3) gives that  $D(h^\gamma) = \varphi(D)h^\gamma \in \mathbb{C}h^\gamma$ . Since it is obvious that  $\varphi \in \mathcal{A}_0^*$ ,  $X(\gamma)$  is an irreducible lowest weight module.

By [10, Theorem 4.5.16] we know that the  $\mathbb{D} \times \mathfrak{g}$ -module  $\mathbb{C}[V]$  has the following decomposition:

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda^+} \text{Hom}_{\mathfrak{g}}(E(\lambda), \mathbb{C}[V]) \otimes_{\mathbb{C}} E(\lambda)$$

where  $\text{Hom}_{\mathfrak{g}}(E(\lambda), \mathbb{C}[V])$  is either (0) or a simple  $\mathbb{D}$ -module (the action being given by  $D(\phi)(x) = D(\phi(x))$  for all  $\phi \in \text{Hom}_{\mathfrak{g}}(E(\lambda), \mathbb{C}[V]), x \in E(\lambda)$ ). Since the  $G$ -module  $V(\tilde{\gamma})$ ,  $\tilde{\gamma} \in \tilde{\Gamma}$ , is isomorphic to  $E(p(\tilde{\gamma}))$ , we obtain that  $\text{Hom}_{\mathfrak{g}}(E(\lambda), \mathbb{C}[V]) = (0)$  if  $\lambda \notin p(\Gamma) = p(\tilde{\Gamma})$ . If  $\lambda \in p(\Gamma)$  the non zero elements of this  $\mathbb{D}$ -module identify with the  $\mathfrak{g}$ -highest weight vectors of weight  $\lambda$  in  $\mathbb{C}[V]$  through the map  $\phi \mapsto \phi(v_\lambda)$ , where  $v_\lambda$  is a highest weight vector in  $E(\lambda)$ . It is easily seen that the  $\mathfrak{g}$ -highest weight vectors of weight  $\lambda$  in  $\mathbb{C}[V]$  are the  $h^{\tilde{\gamma}}$  with  $\tilde{\gamma} = k\tilde{\lambda}_0 + \gamma'$ ,  $p(\gamma') = \lambda$ . Recall that for  $\gamma, \gamma' \in \Gamma$ ,  $p(\gamma) = p(\gamma') \iff \gamma = \gamma'$ ; therefore these  $\mathfrak{g}$ -highest weight vectors are the  $h^{\tilde{\gamma}}$  with  $\tilde{\gamma} \in P_\gamma = \gamma + \mathbb{N}\tilde{\lambda}_0$ , where  $\gamma \in \Gamma$  is such that  $p(\gamma) = \lambda$ . From the previous paragraph we then obtained that  $\text{Hom}_{\mathfrak{g}}(E(\lambda), \mathbb{C}[V]) \cong X(\gamma)$  as  $\mathcal{A}$ -module. The last assertion follows from [10, Theorem 4.5.12].  $\square$

## 5. D-MODULES ON SOME PHV

In this section we continue with a representation  $(\tilde{G} : V)$  of the connected reductive group  $\tilde{G}$  as in §3.1.

**5.1. Representations of Capelli type.** Let

$$\tau : \tilde{\mathfrak{g}} = \text{Lie}(\tilde{G}) \longrightarrow \mathcal{D}(V)$$

be the differential of the  $\tilde{G}$ -action. The elements  $\tau(\xi)$  are linear derivations on  $\mathbb{C}[V]$  given by:

$$\tau(\xi)(\varphi)(v) = \frac{d}{dt} \Big|_{t=0} (e^{t\xi} \cdot \varphi)(v) = \frac{d}{dt} \Big|_{t=0} \varphi(e^{-t\xi} \cdot v),$$

for all  $\varphi \in \mathbb{C}[V], v \in V$ . They are homogeneous of degree zero in the sense that  $[\Theta, \tau(\xi)] = 0$ . The map  $\tau$  yields a homomorphism  $\tau : U(\tilde{\mathfrak{g}}) \rightarrow \mathcal{D}(V)$ .

Recall that the group  $\tilde{G}$  acts naturally on  $\mathcal{D}(V)$ ; the differential of this action is given by  $D \mapsto [\tau(\xi), D]$ ,  $\xi \in \tilde{\mathfrak{g}}, D \in \mathcal{D}(V)$ . Therefore, a subspace  $I \subset \mathcal{D}(V)$  is stable under  $\tilde{G}$ , resp.  $G$ , if and only if  $[\tau(\tilde{\mathfrak{g}}), I] \subset I$ , resp.  $[\tau(\mathfrak{g}), I] \subset I$ . It is then

clear that  $\tilde{\mathbb{D}} = \mathcal{D}(V)^{\tilde{G}} = \{D \in \mathcal{D}(V) : [\tau(\tilde{\mathfrak{g}}), D] = 0\} \subset \mathbb{D} = \mathcal{D}(V)^G = \{D \in \mathcal{D}(V) : [\tau(\mathfrak{g}), D] = 0\}$ . In particular, if  $Z(\tilde{\mathfrak{g}}) = U(\tilde{\mathfrak{g}})^{\tilde{G}}$  is the centre of  $U(\tilde{\mathfrak{g}})$ , then  $\tau(Z(\tilde{\mathfrak{g}})) \subset \tilde{\mathbb{D}}$ .

Following [18, (10.3)] we make the following definition:

**Definition 5.1.** We say that  $(\tilde{G} : V)$  is of Capelli type if:

- $(\tilde{G} : V)$  is irreducible and multiplicity free;
- $\tau(Z(\tilde{\mathfrak{g}})) = \tilde{\mathbb{D}}$ .

**Remarks 5.2.** (1) By Howe and Umeda [18], in the list of  $(\tilde{G} : V)$  which are irreducible and MF we have: three among the thirteen cases are not of Capelli type; two among the ten cases such that  $\dim V//G = 1$  are not of Capelli type. (Thus we are interested in eight cases of the table in Appendix A.)

(2) This definition originates in the case  $(\tilde{G} = \mathrm{GL}(n) \times \mathrm{SL}(n) : V = \mathrm{M}_n(\mathbb{C}))$  where the writing of  $E_0 = f\Delta = \det(x_{ij})\det(\partial_{ij})$  as an element of  $\tau(Z(\tilde{\mathfrak{g}}))$  is the “classical” Capelli identity.

Recall the morphism  $\mathrm{rad} : \mathcal{D}(V)^G \rightarrow \mathcal{D}(V//G)$ . By definition  $\tau(\mathfrak{g})(\mathbb{C}[V]^G) = 0$ , thus one always has:

$$[\mathcal{D}(V)\tau(\mathfrak{g})]^G \subset J = \mathrm{Ker}(\mathrm{rad}).$$

When  $(\tilde{G} : V)$  satisfies Hypothesis B, we have computed in Theorem 4.11 the ideal  $J \subset \mathbb{D}$ :

$$J = \sum_{i=0}^r \mathbb{D}\Omega_i = \sum_{i=0}^r \Omega_i \mathbb{D}$$

where the  $\Omega_i$ ’s are defined in (4.5). When  $(\tilde{G} : V)$  is irreducible we observed in Remark 4.12(1) that we can number these operators so that  $\Omega_r = 0$ , hence  $J = \sum_{i=0}^{r-1} \mathbb{D}\Omega_i$ ; the next proposition gives a more useful description if, moreover,  $(\tilde{G} : V)$  is of Capelli type, i.e. one of the eight cases mentioned in Remark 5.2(1).

**Proposition 5.3.** *Let  $(\tilde{G} : V)$  be of Capelli type and such that  $\dim V//G = 1$ . Then:*

$$\mathrm{Ker}(\mathrm{rad}) = [\mathcal{D}(V)\tau(\mathfrak{g})]^G.$$

*Proof.* In the irreducible case the centre  $C$  of  $\tilde{G}$  acts by scalars on  $V$  and we may assume that:  $\tilde{G} = GC$  with  $C = \mathbb{C}^*$ . Write  $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}$ ,  $\mathfrak{c} = \mathrm{Lie}(C) = \mathbb{C}\zeta$ . Since  $\mathbb{C}[V]^G = \mathbb{C}[f]$  and  $f$  is not  $\tilde{G}$ -invariant one can also suppose that  $\tau(\zeta) = \bar{\Theta} = \frac{1}{n}\Theta$ .

Write  $Z(\tilde{\mathfrak{g}}) = Z(\mathfrak{g})[\zeta]$  and  $Z(\mathfrak{g}) = \mathbb{C} \oplus Z_+(\mathfrak{g})$  where  $Z_+(\mathfrak{g}) = [U(\mathfrak{g})\mathfrak{g}]^G$ . The previous paragraph implies that  $\tau(Z(\tilde{\mathfrak{g}})) = \mathbb{C}[\bar{\Theta}] + \tau(Z_+(\mathfrak{g}))[\bar{\Theta}]$  with  $\tau(Z_+(\mathfrak{g})) = [U(\tau(\mathfrak{g}))\tau(\mathfrak{g})]^G \subset [\mathcal{D}(V)\tau(\mathfrak{g})]^G$ . As recalled above we already know that  $J = \sum_{i=0}^{r-1} \mathbb{D}\Omega_i$ ,  $\Omega_i \in \tilde{\mathbb{D}}$ ,  $0 \leq i \leq r-1$ . By hypothesis  $\tau(Z(\tilde{\mathfrak{g}})) = \tilde{\mathbb{D}}$ , thus we can write each  $\Omega_j$  as follows:

$$\Omega_j = \sum_{k \geq 0} \bar{\Theta}^k P_{j,k}, \quad P_{j,k} = p_{j,k} + P'_{j,k}, \quad p_{j,k} \in \mathbb{C}, \quad P'_{j,k} \in \tau(Z_+(\mathfrak{g})).$$

Recall that  $[\mathcal{D}(V)\tau(\mathfrak{g})]^G \subset J$ ; thus we have  $\mathrm{rad}(P'_{j,k}) = 0$  and we obtain:  $\mathrm{rad}(\Omega_j) = \sum_{k \geq 0} \theta^k p_{j,k} = 0$  in  $R = \mathbb{C}[z, \delta, \theta]$ . Therefore  $p_{j,k} = 0$  for all  $k \geq 0$ , which gives  $\Omega_j = \sum_{k \geq 0} \bar{\Theta}^k P'_{j,k} \in [\mathcal{D}(V)\tau(\mathfrak{g})]^G$  and  $J \subset [\mathcal{D}(V)\tau(\mathfrak{g})]^G$ .  $\square$

**5.2. Application to  $D$ -modules.** *In this subsection we assume that  $(\tilde{G} : V)$  satisfies Hypothesis B, hence  $(\tilde{G} : V)$  is MF,  $\mathbb{C}[V]^G = \mathbb{C}[f]$ ,  $f \notin \mathbb{C}[V]^{\tilde{G}}$ .*

Recall from Theorem 4.11 that  $\mathbb{D} = \bigoplus_{k \in \mathbb{Z}} \mathbb{D}[k]$ . We have seen (Lemma 4.10) that if  $D \in \tilde{\mathbb{D}}$  there exists a polynomial  $b_D(s)$  such that  $D = b_D(\bar{\Theta}) + D_1$ ,  $D_1 \in J = \mathrm{Ker}(\mathrm{rad})$ .

Fix  $P \in \mathbb{D}[k]$  and write  $P = DQ_k$ ,  $Q_k = f^k$  if  $k \geq 0$ ,  $Q_k = \Delta^{-k}$  if  $k < 0$ ,  $D \in \widetilde{\mathbb{D}}$ . Then:

$$P = b_D(\bar{\Theta})Q_k + P_1, \quad P_1 = D_1Q_k \in J. \quad (5.1)$$

Observe that, since  $\Delta(f^m) = b^*(m)f^{m-1}$ ,

$$P(f^m) = \begin{cases} b_D(m+k)f^{m+k} & \text{if } k \geq 0; \\ b^*(m)b^*(m-1) \cdots b^*(m+k+1)b_D(m+k)f^{m+k} & \text{if } k < 0. \end{cases}$$

Therefore if we set  $a_P(s) = b_D(s+k)$  if  $k \geq 0$ , or  $a_P(s) = b_D(s+k) \prod_{j=0}^{k+1} b^*(s+j)$  if  $k < 0$ , then  $\deg a_P = \deg b_P$  or  $\deg b_P + nk$ , and  $P(f^m) = a_P(m)f^{m+k}$ . Notice that  $a_{Q_k}(s) = 1$  if  $k \geq 0$ ,  $a_{Q_k}(s) = \prod_{j=0}^{k+1} b^*(s+j)$  if  $k < 0$ , thus  $a_P(s) = b_D(s+k)a_{Q_k}(s)$ .

When  $(\tilde{G} : V) = (\mathrm{GL}(n) : S^2\mathbb{C}^n)$ , similar results were obtained by Masakazu Muro in [36, Proposition 3.8], where our polynomial  $a_P(s)$  is denoted by  $b_P(s)$ .

**Definition 5.4.** Let  $P \in \mathbb{D}[k]$  be as above and define the  $D$ -module associated to  $P$  by:

$$\mathcal{M}_P = \mathcal{D}(V)/(\mathcal{D}(V)\tau(\mathfrak{g}) + \mathcal{D}(V)P) = \mathcal{D}(V)/I_P,$$

where  $I_P = \mathcal{D}(V)\tau(\mathfrak{g}) + \mathcal{D}(V)P$ .

*Remark.* Let  $I \subset \mathcal{D}(V)$  be a left ideal containing  $\mathcal{D}(V)\tau(\mathfrak{g})$ . Since the condition  $[\tau(\mathfrak{g}), I] \subset I$  is satisfied, the group  $G$  acts naturally on  $I$  and therefore on  $M = \mathcal{D}(V)/I$ . Furthermore, the differential of this action is given by the left multiplication on  $M$  by  $\tau(\xi)$ ,  $\xi \in \mathfrak{g}$ . It follows, see [17, §II.2, Theorem], that  $M$  is a  $G$ -equivariant  $D$ -module on  $V$  (cf. [17] for the definition). This is in particular true for  $\mathcal{M}_P$ .

Following [33, 34, 35, 36] we want to study the solutions of the differential system associated to  $\mathcal{M}_P$ . We first need to study the characteristic variety on  $\mathcal{M}_P$ .

Recall that  $\mathcal{D}(V)$  is filtered by the order of differential operators, see [4, 16], and that the associated graded ring of  $\mathcal{D}(V)$  identifies with  $\mathbb{C}[T^*V]$ , where  $T^*V = V \times V^*$  is the cotangent bundle of  $V$ . If  $u \in \mathcal{D}(V)$  we denote its order by  $\mathrm{ord} u$  and its principal symbol by  $\sigma(u) \in \mathbb{C}[T^*V] = S(V^*) \otimes S(V)$ . Let  $M$  be a finitely generated  $\mathcal{D}(V)$ -module, then one can endow  $M$  with a good filtration and one defines the characteristic variety of  $M$ , denoted by  $\mathrm{Ch} M$ , as being the support in  $T^*M$  of the associated graded module (see, e.g., [16, §I.3]). When  $M = \mathcal{D}(V)/I$ ,  $\mathrm{Ch} M \subset T^*M$  is the variety of zeroes of the symbols  $\sigma(u)$ ,  $u \in I$ . Recall that if  $\dim \mathrm{Ch} M \leq \dim V$  the  $D$ -module  $M$  is called holonomic (one always have  $\dim \mathrm{Ch} M \geq \dim V$  if  $M \neq (0)$ ).

*Remark.* Let  $v \in V, v^* \in V^*, \xi \in \tilde{\mathfrak{g}}$ . One has:

$$\sigma(\Theta)(v, v^*) = \langle v^*, v \rangle, \quad \sigma(\tau(\xi))(v, v^*) = -\langle \xi.v^*, v \rangle = \langle v^*, \xi.v \rangle.$$

**Lemma 5.5.** Let  $k \in \mathbb{Z}$  and  $P = DQ_k$  be as above.

- (a) There exists  $Q'_k \in J$  and  $q_k(s) \in \mathbb{C}[s]$  such that  $Q_k Q_{-k} = q_k(\bar{\Theta}) + Q'_k$ ,  $\mathrm{ord}(Q_k Q_{-k}) = \mathrm{ord}(Q_{-k} Q_k) = \deg q_k = |k|n$ .
- (b) Write  $P = b_D(\bar{\Theta})Q_k + P_1$  as in (5.1) and set  $P_0 = b_D(\bar{\Theta})Q_k$ . Then  $P_0 Q_{-k} = b_D(\bar{\Theta})q_k(\bar{\Theta}) + P_2$  with  $P_2 \in J$  and  $\mathrm{ord} P_2 \leq \mathrm{ord}(P_0 Q_{-k}) = \deg(b_D q_k)$ .

*Proof.* (a) Let  $m \in \mathbb{N}$ . Then:

$$Q_k Q_{-k}(f^m) = a_{Q_{-k}}(m)Q_k(f^{m-k}) = a_{Q_{-k}}(m)a_{Q_k}(m-k)f^m.$$

Set  $q_k(s) = a_{Q_{-k}}(s)a_{Q_k}(s-k) \in \mathbb{C}[s]$ . The previous computation yields  $Q_k Q_{-k} = q_k(\bar{\Theta}) + Q'_k$  with  $Q'_k \in J$ . Since  $\deg a_{Q_k} = 1$  or  $-nk$  (if  $k \geq 0$  or  $< 0$ ) we get that  $\deg q_k = |k|n$ . On the other hand,  $Q_k Q_{-k} = f^k \Delta^{-k}$  or  $\Delta^k f^{-k}$  has order  $\mathrm{ord} \Delta^{|k|}$ , i.e.  $|k|n = \deg q_k$ . This implies in particular that  $\mathrm{ord} Q'_k \leq \mathrm{ord}(Q_k Q_{-k})$ .

(b) From (a) we obtain  $P_0 Q_{-k} = b_D(\bar{\Theta})q_k(\bar{\Theta}) + P_2$ ,  $P_2 = b_D(\bar{\Theta})Q'_k \in J$ . One has:  $\text{ord}(P_0 Q_{-k}) = \text{ord} P_0 + \text{ord} Q_{-k} = \deg b_D + \text{ord} Q_k + \text{ord} Q_{-k} = \deg b_D + \text{ord}(Q_k Q_{-k})$ . Therefore,  $\text{ord} P_2 = \deg b_D + \text{ord} Q'_k \leq \deg b_D + \text{ord}(Q_k Q_{-k}) = \text{ord}(P_0 Q_{-k})$ , as desired.  $\square$

As in [45, Section 3] define the commuting varieties of  $(\tilde{G} : V)$  and  $(G : V)$  by:

$$\tilde{\mathcal{C}}(V) = \{(v, v^*) \in T^*V : \langle v^*, \tilde{\mathbf{g}}.v \rangle = 0\}, \quad \mathcal{C}(V) = \{(v, v^*) \in T^*V : \langle v^*, \mathbf{g}.x \rangle = 0\}.$$

Recall that  $(\tilde{G} : V)$  is MF; this implies [21] that  $\tilde{G}$  has finitely many orbits in  $V$ , denoted by  $O_1, \dots, O_t$ . Set  $T_{O_i}^*V = \{(v, v^*) \in T^*V : v \in O_i, \langle v^*, \tilde{\mathbf{g}}.v \rangle = 0\}$ . By [47], see also [45, Theorem 3.2], we have the following result:

**Theorem 5.6.** *The irreducible components of  $\tilde{\mathcal{C}}(V)$  are the closures of the conormal bundles of the orbits, i.e. the  $\mathcal{C}_i = \overline{T_{O_i}^*V}$ . In particular,  $\tilde{\mathcal{C}}(V)$  is equidimensional of dimension  $\dim V$ .*

**Remark 5.7.** Set  $\mathcal{C}(V)' = \{(v, v^*) \in T^*V : \sigma(u)(v, v^*) = 0 \text{ for all } u \in \mathcal{D}(V)\tau(\mathbf{g})\}$ . Thus  $\mathcal{C}(V)'$  is the characteristic variety of the  $\mathcal{D}(V)$ -module

$$\mathcal{N} = \mathcal{D}(V)/\mathcal{D}(V)\tau(\mathbf{g}) \quad (5.2)$$

Let  $P \in \mathbb{D}[k]$ . Then we clearly have:

$$\tilde{\mathcal{C}}(V) \subset \mathcal{C}(V), \quad \text{Ch } \mathcal{M}_P \subset \mathcal{C}(V)' \subset \mathcal{C}(V).$$

We will now assume that  $(\tilde{G} : V)$  is irreducible. This means that  $(\tilde{G} : V)$  is one of the cases (1) to (10) in the table of Appendix A. We may assume here that  $\tilde{G} = GC$ ,  $C \cong \mathbb{C}^*$ . We then write  $\tilde{\mathbf{g}} = \mathbf{g} \oplus \mathbb{C}\zeta$ ,  $\mathfrak{c} = \text{Lie}(C) = \mathbb{C}\zeta$  where  $\zeta$  is chosen such that  $\tau(\zeta) = \bar{\Theta}$  (i.e.  $\zeta$  acts as  $\frac{1}{n}\text{id}_V$  on  $V$ ).

**Corollary 5.8.** *Under the previous hypothesis:*

- $\dim \mathcal{C}(V) = \dim V + 1$ ;
- the module  $\mathcal{N}$  is not holonomic, i.e.  $\dim \mathcal{C}(V)' = \dim V + 1$ .

*Proof.* By [21, Theorem 1] the representation  $(G : V)$  is visible, i.e.  $\{v \in V : f(v) = 0\}$  contains a finite number of  $G$ -orbits. Then, since  $\mathbb{C}(V)^G = \mathbb{C}(f)$  and  $f$  is non constant, [44, Theorems 2.3 & 3.1, Corollary 2.5] yield  $\dim \mathcal{C}(V) = \dim V + 1$ .

Recall that if  $M$  is any  $\mathcal{D}(V)$ -module one can identify  $\text{Hom}_{\mathcal{D}(V)}(\mathcal{N}, M)$  with the space  $\{x \in M : \tau(\mathbf{g}).x = 0\}$ . In particular,  $\text{Hom}_{\mathcal{D}(V)}(\mathcal{N}, \mathbb{C}[V]) \cong \mathbb{C}[V]^G = \mathbb{C}[f]$ . If  $\mathcal{N}$  were holonomic we would have  $\dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}(V)}(\mathcal{N}, \mathbb{C}[V]) < \infty$ , cf. [32, Theorem 9.5.5], which is absurd.  $\square$

Assume that  $(\tilde{G} : V)$  is of Capelli type and let  $P = b_D(\bar{\Theta})Q_k + P_1$ ,  $P_1 \in J$ , as in (5.1). By Proposition 5.3,  $I_P = \mathcal{D}(V)\tau(\mathbf{g}) + \mathcal{D}(V)b_D(\bar{\Theta})Q_k$ . Therefore:

$$\mathcal{M}(b_D, k) = \mathcal{M}_P = \mathcal{D}(V)/(\mathcal{D}(V)\tau(\mathbf{g}) + \mathcal{D}(V)b_D(\bar{\Theta})Q_k) \quad (5.3)$$

depends only on the polynomial  $b_D(s)$  and the integer  $k$ . We need to know in which case  $\mathcal{M}(b_D, k)$  is holonomic.

**Theorem 5.9.** *Assume that  $(\tilde{G} : V)$  is of Capelli type and let  $P = b_D(\bar{\Theta})Q_k + P_1$  with  $P_1 \in \text{Ker}(\text{rad})$ . The following are equivalent:*

- (i)  $b_D(s) \neq 0$ ;
- (ii)  $\mathcal{M}(b_D, k) = \mathcal{M}_P$  is holonomic.

*In this case  $\text{Ch } \mathcal{M}_P \subset \tilde{\mathcal{C}}(V)$  is a union of  $\mathcal{C}_i$ 's.*

*Proof.* Suppose that  $b_D = 0$ , then  $\mathcal{M}(b_D, k) = \mathcal{N}$  is not holonomic. Conversely, suppose  $b_D \neq 0$ . We are going to show that  $\mathcal{M}_P \subset \tilde{\mathcal{C}}(V)$ , then Theorem 5.6 will give the result.

Set  $\alpha = \sigma(\bar{\Theta}) = \sigma(\tau(\zeta)) \in \mathbb{C}[T^*V]$ , hence  $\alpha(v, v^*) = \frac{1}{n}\langle v^*, v \rangle$ . Since  $\tilde{\mathbf{g}} = \mathbf{g} \oplus \mathbb{C}\zeta$ , we have  $\tilde{\mathcal{C}}(V) = \mathcal{C}(V) \cap \alpha^{-1}(0)$ . Using the notation of Lemma 5.5 we set  $P_0 = b_D(\bar{\Theta})Q_k$ ,  $P_0Q_{-k} = b_D(\bar{\Theta})q_k(\bar{\Theta}) + P_2$  with  $P_2 \in J$ . Notice that  $\mathcal{M}_P = \mathcal{M}_{P_0}$ . Since  $b_D \neq 0$ , we can write  $h(s) = b_D(s)q_k(s) = h_d s^d + h_{d-1} s^{d-1} + \dots$ ,  $d = \deg h(s) = \deg b_D(s) + \deg q_k(s) = \deg b_D(s) + |k|n \geq 0$ . Recall that  $\text{ord } P_2 \leq \text{ord } P_0Q_k = d$ , therefore  $\sigma(P_0Q_k) = \sigma(h(\bar{\Theta}))$  or  $\sigma(h(\bar{\Theta})) + \sigma(P_2)$  (depending on  $\text{ord } P_2 < d$  or  $\text{ord } P_2 = d$ ).

If  $d = 0$ , we get  $k = 0$ ,  $b_D \in \mathbb{C}^*$ , thus  $\mathcal{M}_P = (0)$  and the claim is obvious. Suppose  $d \geq 1$  and let  $(v, v^*) \in \text{Ch } \mathcal{M}_P \subset \text{Ch } \mathcal{N} = \mathcal{C}(V)'$ . From  $P_2 \in J = [\mathcal{D}(V)\tau(\mathbf{g})]^G \subset \mathcal{D}(V)\tau(\mathbf{g})$  we know that  $\sigma(P_2)(v, v^*) = 0$ . Therefore,  $\sigma(P_0)(v, v^*) = 0$  implies

$$\begin{aligned} 0 &= \sigma(P_0)(v, v^*)\sigma(Q_{-k})(v, v^*) = \sigma(P_0Q_{-k})(v, v^*) = \sigma(h(\bar{\Theta}))(v, v^*) \\ &= h_d\sigma(\bar{\Theta})^d(v, v^*) = h_d\alpha(v, v^*)^d. \end{aligned}$$

Hence  $\alpha(v, v^*) = 0$  and this proves  $(v, v^*) \in \mathcal{C}(V)' \cap \alpha^{-1}(0) \subset \tilde{\mathcal{C}}(V)$ , as desired.  $\square$

**Remark 5.10.** The previous result generalizes the main step in the proof of [36, Theorem 4.1] (see also [36, Theorem 6.1 and Remark 6.1]). Indeed the homogeneous elements of degree  $kn$  in [36] are the elements of  $\mathbb{D}[k]$  and [36, Lemma 4.1] shows, when  $(\tilde{G} : V) = (\text{GL}(n) : S^2\mathbb{C}^n)$ , that  $\mathcal{M}_P$  is holonomic when  $P \neq 0$  and  $\deg a_P = \text{ord } P$ . Since  $a_P(s) = b_D(s+k)a_{Q_k}(s) \neq 0 \iff b_D(s) \neq 0$ , Theorem 5.9 ensures that a more general result holds for representations of Capelli type; notice that the example given [36, Remark 4.1] is  $P = \Omega_0 = f\Delta - b^*(\bar{\Theta}) \in J$ , hence  $\mathcal{M}_P = \mathcal{N}$  is not holonomic. Observe also that Theorem 5.9 is proved in [33, Proposition 2.1] in the case  $(b_D = 1, k \geq 0)$ .

**5.3. Solutions of invariant differential equations.** We continue with a representation  $(\tilde{G} : V)$  of Capelli type. Let  $(\tilde{G}_{\mathbb{R}} : V_{\mathbb{R}})$  be a real form of  $(\tilde{G} : V)$  in the sense of [25, §4.1, Proposition 4.1], cf. also [33, §1.2] and [13, §4.2 & §4.3]. Such a form always exists. Let  $M$  be a finitely generated  $\mathcal{D}(V)$ -module of the form  $\mathcal{D}(V)/I$ . Denote by  $\mathcal{B}_{V_{\mathbb{R}}}$  the space of hyperfunctions on  $V_{\mathbb{R}}$  (see, e.g., [13, §4.1]). Then  $\mathcal{B}_{V_{\mathbb{R}}}$  is a  $\mathcal{D}(V)$ -module and the “hyperfunction solutions space” of  $M$  is:

$$\text{Sol}(M, \mathcal{B}_{V_{\mathbb{R}}}) = \{T \in \mathcal{B}_{V_{\mathbb{R}}} : D.T = 0 \text{ for all } D \in I\} \equiv \text{Hom}_{\mathcal{D}_V}(M, \mathcal{B}_{V_{\mathbb{R}}}).$$

Notice that since any distribution on  $V_{\mathbb{R}}$  can be viewed as a hyperfunction, the “distribution solutions space” of  $M$  is contained in  $\text{Sol}(M, \mathcal{B}_{V_{\mathbb{R}}})$ . An element  $T$  of  $\mathcal{B}_{V_{\mathbb{R}}}$  such that  $\tau(\mathbf{g}).T = 0$  is said to be  $\mathbf{g}$ -invariant; it is called quasi-homogeneous if there exist  $t \in \mathbb{N}, \mu \in \mathbb{C}$  such that  $(\Theta - \mu)^t.T = 0$ . Assume that  $I = \mathcal{D}(V)\tau(\mathbf{g}) + \mathcal{D}(V)P$  for some  $P \in \mathcal{D}(V)$ , then  $\text{Sol}(M, \mathcal{B}_{V_{\mathbb{R}}})$  identifies with the space of  $\mathbf{g}$ -invariant hyperfunctions  $T$  which are solutions of the equation  $P.T = 0$ .

The next corollary has been proved by M. Muro [36, Theorem 4.1] for the real form  $(\tilde{G}_{\mathbb{R}} = \text{GL}(n, \mathbb{R}), V_{\mathbb{R}} = \text{Sym}_n(\mathbb{R}))$  of  $(\tilde{G} : V) = (\text{GL}(n) : S^2\mathbb{C}^n)$  when  $P \neq 0$  and  $\deg a_P = \text{ord } P$ . (See Remark 5.10 for more details.)

**Corollary 5.11.** *Let  $(\tilde{G} : V)$  be of Capelli type and let  $(\tilde{G}_{\mathbb{R}} : V_{\mathbb{R}})$  be a real form of  $(\tilde{G} : V)$ . Let  $P \in \mathbb{D}[k]$  and write  $P = b_D(\bar{\Theta})Q_k + P_1$ ,  $P_1 \in \text{Ker}(\text{rad})$ , as in (5.1). Assume that  $b_D \neq 0$ . Then,  $\text{Sol}(P, \mathcal{B}_{V_{\mathbb{R}}}) = \text{Sol}(\mathcal{M}_P, \mathcal{B}_{V_{\mathbb{R}}}) = \{T \in \mathcal{B}_{V_{\mathbb{R}}} : T \text{ } \mathbf{g}\text{-invariant, } P.T = 0\}$  is finite dimensional and has a basis of quasi-homogeneous elements; it depends only on the polynomial  $b_D(s)$  and the integer  $k$ .*

*Proof.* We merely repeat the proof of M. Muro (loc. cit.). A well-known result of M. Kashiwara (see [23, Théorème 5.1.6]) says that if  $M$  holonomic  $\text{Sol}(M, \mathcal{B}_{V_{\mathbb{R}}})$  is a finite dimensional  $\mathbb{C}$ -vector space. Therefore by combining the remarks above and Theorem 5.9 we obtain that  $\mathcal{S} = \text{Sol}(P, \mathcal{B}_{V_{\mathbb{R}}})$  is finite dimensional. Now observe

that  $[\Theta, P] = kP$  and  $[\Theta, \tau(\mathfrak{g})] = 0$  imply that  $\mathcal{S}$  is stable under the action of  $\Theta$ . Therefore  $\mathcal{S}$  decomposes as a direct sum of spaces of the form  $\text{Ker}(\Theta - \mu \text{id}_{\mathcal{S}})^t$ . We have noticed after (5.3) that  $\mathcal{M}_P$  depends only on  $b_D(s)$  and  $k$ , therefore it is also the case for  $\mathcal{S} = \text{Hom}_{\mathcal{D}_V}(\mathcal{M}_P, \mathcal{B}_{V_{\mathbb{R}}})$ .  $\square$

**5.4. Regular holonomic modules.** Assume that  $(\tilde{G} : V)$  is of Capelli type and  $\dim V // G = 1$ .

We filter  $\mathcal{D}(V)$  by order of differential operators and we set  $\mathcal{D}(V)_j = \{D \in \mathcal{D}(V) : \text{ord } D \leq j\}$ . For sake of completeness we now recall some known (or easy) results.

**Definition 5.12.** Let  $M$  be a finitely generated  $\mathcal{D}(V)$ -module.

- $M$  is *monodromic* if  $\dim_{\mathbb{C}} \mathbb{C}[\Theta].x < \infty$  for all  $x \in M$ .
- $M$  is *homogeneous* if there exists a  $\Theta$ -stable good filtration on  $M$ , i.e. a good filtration  $FM = (F_p M)_{p \in \mathbb{N}}$  such that  $\Theta.F_p M \subset F_p M$  for all  $p$ .
- $x \in M$  is *quasi-homogeneous* (of weight  $\lambda$ ) if there exists  $j \in \mathbb{N}$  and  $\lambda \in \mathbb{C}$  such that  $(\Theta - \lambda)^j.x = 0$ ; we set  $M_{\lambda} = \bigcup_{j \in \mathbb{N}} \text{Ker}_M(\Theta - \lambda)^j$ .

Recall the following result proved in [39, Theorem 1.3] (which holds in the analytic case).

**Theorem 5.13.** *Let  $M$  be a homogeneous  $\mathcal{D}(V)$ -module. Then:*

- (1)  $M = \mathcal{D}(V).E$ ,  $E$  finite dimensional and generated by quasi-homogeneous elements;
- (2) if  $FM = (F_p M)_{p \in \mathbb{N}}$  is a  $\Theta$ -stable good filtration on  $M$ , the space  $F_p M \cap M_{\lambda}$  is finite dimensional for all  $p \in \mathbb{N}$ ,  $\lambda \in \mathbb{C}$ .

Using this result it is not difficult to obtain the next corollary.

**Corollary 5.14.** *Let  $M$  be a finitely generated  $\mathcal{D}(V)$ -module. The following assertions are equivalent:*

- (i)  $M$  is monodromic;
- (ii)  $M$  is homogeneous;
- (iii)  $M = \mathcal{D}(V).E$ ,  $E$  finite dimensional such that  $\Theta.E \subset E$ .

Recall that a holonomic  $\mathcal{D}(V)$ -module is regular if there exists a good filtration  $FM$  on  $M$  such that the ideal  $\text{ann}_{\mathbb{C}[T^*V]} \text{gr}^F(M)$  is radical, see [24, Corollary 5.1.11]. Denote by:

- $\text{mod}_{\tilde{\mathcal{C}}}^{\text{rh}}(\mathcal{D}_V)$  the category of regular holonomic whose characteristic variety is contained in  $\tilde{\mathcal{C}}(V)$ ;
- $\text{mod}_{\tilde{\mathcal{C}}}^{\Theta}(\mathcal{D}_V)$  the category of monodromic modules with characteristic variety contained in  $\tilde{\mathcal{C}}(V)$ .

Let  $G_1$  be the simply connected cover of  $G$  and set  $\tilde{G}_1 = G_1 \times C$  (recall that  $C \cong \mathbb{C}^*$  is the connected component of the centre of  $\tilde{G}$ ). One has:  $\text{Lie}(\tilde{G}_1) = \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}$ ,  $\mathfrak{c} = \mathbb{C}\zeta$ , where  $\tau(\zeta) = \bar{\Theta}$  as above. The group  $\tilde{G}_1$  maps onto  $G \times C$ , and therefore onto  $\tilde{G} = GC$ . It follows that  $\tilde{G}_1$  and  $G \times C$  act on  $V$ ; the orbits in  $V$  then coincide with the  $\tilde{G}$ -orbits  $O_1, \dots, O_t$ . The category of  $\tilde{G}_1$ -equivariant  $\mathcal{D}(V)$ -modules is denoted by

$$\text{mod}^{\tilde{G}_1}(\mathcal{D}_V).$$

Observe that if

$$\text{mod}^{G \times C}(\mathcal{D}_V)$$

is the category of  $(G \times C)$ -equivariant  $\mathcal{D}(V)$ -modules, any object in  $\text{mod}^{G \times C}(\mathcal{D}_V)$  can be naturally considered as an object of  $\text{mod}^{\tilde{G}_1}(\mathcal{D}_V)$ . When  $G$  is simply connected, e.g.  $G = \text{SL}(n)$ ,  $\text{SL}(n) \times \text{SL}(n)$ ,  $\text{SL}(n) \times \text{Sp}(m)$  or  $\text{G}_2$ , we have  $\tilde{G}_1 = G \times C$  and these two categories are the same.

**Lemma 5.15.** *Let  $M$  be a finitely generated  $\mathcal{D}(V)$ -module. Then:*

$$(i) \ M \in \mathbf{mod}^{\tilde{G}_1}(\mathcal{D}_V) \iff (ii) \ M \in \mathbf{mod}_{\tilde{C}}^{\mathrm{rh}}(\mathcal{D}_V) \implies (iii) \ M \in \mathbf{mod}_{\tilde{C}}^{\Theta}(\mathcal{D}_V).$$

*In particular,  $\mathbf{mod}^{G \times C}(\mathcal{D}_V) = \mathbf{mod}_{\tilde{C}}^{\mathrm{rh}}(\mathcal{D}_V)$  when  $G$  is simply connected.*

*Proof.* (i)  $\Rightarrow$  (ii): By [4, Theorem 12.11], or [17, §5],  $M$  is regular holonomic. Its characteristic variety  $\mathrm{Ch} M$  is therefore a  $\tilde{G}$ -stable subvariety of  $T^*V$ . Let  $X$  be an irreducible component of  $\mathrm{Ch} M$ ; then  $X$  is a Lagrangian conical closed  $\tilde{G}$ -stable subvariety of  $T^*V$  and, if  $\pi : T^*V \rightarrow V$  is the natural projection, [22, § 5, Lemme 1], implies that  $X = \overline{T_{\pi(X)^{\mathrm{reg}}}^* V}$ . But it is easy to see that  $\pi(X)^{\mathrm{reg}}$  (the smooth locus of  $\pi(X)$ ) is equal to  $O_j$  for some  $1 \leq j \leq t$ . Hence  $X = C_j$  and  $\mathrm{Ch} M \subset \tilde{C}(V)$ .

(ii)  $\Rightarrow$  (iii) and (i): (We mimic the proof of [39, Proposition 1.6].) Choose a good filtration such that  $I(M) = \mathrm{ann}_{\mathbb{C}[T^*V]} \mathrm{gr}^F(M)$  is radical. Since  $\mathrm{Ch} M \subset \tilde{C}(V)$ , the principal symbols  $\alpha = \sigma(\tilde{\Theta})$  and  $\sigma(\tau(\xi))$ ,  $\xi \in \mathfrak{g}$ , belong to  $I(M)$ , that is to say  $\sigma(\tau(\xi)) \mathrm{gr}_j^F(M) = \alpha \mathrm{gr}_j^F(M) = (0) \subset \mathrm{gr}_{j+1}^F(M)$ ; in other words:  $\tilde{\Theta}.F_j M = \Theta.F_j M \subset F_j M$  and  $\tau(\xi).F_j M \subset F_j M$ . In particular,  $M$  is homogeneous, i.e. monodromic. Let  $x \in M$ . Since  $\dim \mathbb{C}[\Theta].x < \infty$  there exist  $j \in M$  and  $\lambda_1, \dots, \lambda_l$  such that  $x \in \sum_{i=1}^l F_j M \cap M_{\lambda_i}$ . From  $[\tau(\mathfrak{g}), \Theta] = 0$  and  $\tau(\mathfrak{g}).F_j M \subset F_j M$  it follows that  $\tau(\mathfrak{g}).F_j M \cap M_{\lambda_i} \subset F_j M \cap M_{\lambda_i}$ ; hence  $U(\mathfrak{g}).x$  is contained in the finite dimensional space  $\sum_{i=1}^l F_j M \cap M_{\lambda_i}$ , cf. Theorem 5.13. This shows that the action of  $\tilde{\mathfrak{g}}$  on  $M$  given by the  $\tau(\xi)$ ,  $\xi \in \tilde{\mathfrak{g}}$ , is locally finite. The formula

$$e^{t\xi}.x = \exp(t\tau(\xi)).x = \sum_{k \geq 0} \frac{t^k}{k!} \tau(\xi)^k .x$$

then yields a rational action of  $\tilde{G}_1$  on  $M$  whose differential is given by multiplication by the elements  $\tau(\xi)$ . It remains to check that this action is compatible with the action of  $\tilde{G}$  on  $\mathcal{D}(V)$ , which is an easy exercise.  $\square$

Following [39, 40, 41, 42, 43] we want to describe the category  $\mathbf{mod}_{\tilde{C}}^{\mathrm{rh}}(\mathcal{D}_V)$  in terms of a category of modules over  $U = R = \mathbb{C}[z, \delta, \theta]$ . A finitely generated  $U$ -module  $N$  is called *monodromic* if, for all  $v \in N$ ,  $\dim_{\mathbb{C}} \mathbb{C}[\theta].v < \infty$ . Denote by

$$\mathbf{mod}^{\theta}(U)$$

the category of monodromic modules. Observe that  $N \in \mathbf{mod}^{\theta}(U)$  decomposes as

$$N = \bigoplus_{\lambda \in \mathbb{C}} N_{\lambda}, \quad N_{\lambda} = \bigcup_{j \geq 0} \mathrm{Ker}_N(\theta - \lambda)^j.$$

Recall that  $[\theta, z] = z$ ,  $[\theta, \delta] = -\delta$ ,  $z\delta = b^*(\theta)$ ,  $\delta z = b(\theta)$  and that the roots of  $b(-s)$  are  $\lambda_0 + 1 = 1, \lambda_1 + 1, \dots, \lambda_{n-1} + 1$ , cf. Theorem 3.1. We then obtain:

- $z.N_{\lambda} \subset N_{\lambda+1}$ ,  $\delta.N_{\lambda} \subset N_{\lambda-1}$ ;
- $\dim N_{\lambda} < \infty$  ( $N$  is finitely generated);
- $z\delta$ , resp.  $\delta z$ , is bijective on  $N_{\lambda}$  if and only if  $\lambda \neq -\lambda_j$ , resp.  $\lambda \neq -(\lambda_j + 1)$ , therefore  $z : N_{\lambda} \xrightarrow{\sim} N_{\lambda+1}$ ,  $\delta : N_{\lambda+1} \xrightarrow{\sim} N_{\lambda}$  if  $\lambda \neq -(\lambda_j + 1)$ .

From these properties it is easy to give a description of the category  $\mathbf{mod}^{\theta}(U)$  in terms of “finite diagrams of linear maps” as in [39, 40, 41, 43].

Let  $M \in \mathbf{mod}^{G \times C}(\mathcal{D}_V)$ . Since the differential of the  $G$ -action on  $M$  is given by  $x \mapsto \xi.x$ ,  $\xi \in \mathfrak{g}$ ,  $x \in M$ , one has:

$$\Phi M = M^G = \{x \in M : \tau(\mathfrak{g}).x = 0\}. \quad (5.4)$$

Therefore  $[\mathcal{D}(V)\tau(\mathfrak{g})]^G.M = 0$  and, via  $\mathrm{rad} : \mathcal{D}(V)^G/[\mathcal{D}(V)\tau(\mathfrak{g})]^G \xrightarrow{\sim} U$  (Theorem 3.9),  $\Phi M$  can be considered as a  $U$ -module by:  $u.x = D.x$  if  $u = \mathrm{rad}(D)$ ,  $x \in M^G$ . Observe also that the isomorphism  $\mathrm{rad}$  yields a natural structure of

right  $U$ -module on the module  $\mathcal{N} = \mathcal{D}(V)/\mathcal{D}(V)\tau(\mathfrak{g})$  by:  $\bar{a}.u = \overline{aD}$  if  $\bar{a} \in \mathcal{N}$  and  $u = \text{rad}(D) \in U$ . For  $N \in \text{mod}^\theta(U)$  we set:

$$\Psi N = \mathcal{N} \otimes_U N. \quad (5.5)$$

With these notation we have:

**Proposition 5.16.** (1) *Let  $M \in \text{mod}^{G \times C}(\mathcal{D}_V)$  and  $N \in \text{mod}^\theta(U)$ , then:*

$$\Phi M \in \text{mod}^\theta(U), \quad \Psi N \in \text{mod}^{G \times C}(\mathcal{D}_V), \quad \Phi \Psi N = N.$$

(2) *Suppose that any  $M \in \text{mod}^{G \times C}(\mathcal{D}_V)$  is generated by  $M^G$  as a  $\mathcal{D}(V)$ -module. Then the categories  $\text{mod}^{G \times C}(\mathcal{D}_V)$  and  $\text{mod}^\theta(U)$  are equivalent via the functors  $\Phi$  and  $\Psi$ . If furthermore  $G$  is simply connected, we obtain:  $\text{mod}_C^{\text{rh}}(\mathcal{D}_V) \equiv \text{mod}^\theta(U)$ .*

*Proof.* (1) From  $G$  reductive and  $M$  finitely generated, one deduces that the  $\mathcal{D}(V)^G$ -module  $M^G$  is finitely generated. Recall that  $M$  is monodromic (Lemma 5.15); since  $\theta.x = \text{rad}(\bar{\Theta}).x = \bar{\Theta}.x$  it follows that  $\Phi M$  is monodromic. Thus  $\Phi N \in \text{mod}^\theta(U)$ .

It is clear that  $\Psi N$  is finitely generated over  $\mathcal{D}(V)$ . The group  $G$  acts naturally on  $\mathcal{D}(V)$  and this action passes to  $\mathcal{N}$  (note that  $\mathcal{D}(V)\tau(\mathfrak{g})$  is  $G$ -stable). One easily checks that one can endow  $\mathcal{N} \otimes_U N$  with a rational  $G$ -module structure by setting:  $g.(\bar{a} \otimes_U x) = \overline{g.a} \otimes_U x$  for  $\bar{a} \in \mathcal{N}, g \in G, x \in N$ . Notice that since  $N$  is monodromic the group  $C = \exp(\mathbb{C}\zeta)$  acts on  $N$  by  $e^{t\zeta}.x = \exp(t\theta).x$ . One can then verify that  $C$  acts on  $\mathcal{N} \otimes_U N$  by:  $e^{t\zeta}.(\bar{a} \otimes_U x) = e^{t\zeta}.\bar{a} \otimes_U \exp(t\theta).x$ ,  $t \in \mathbb{C}$ . One shows without difficulty that this  $G \times C$ -action is compatible with the  $\tilde{G}$ -action on  $\mathcal{D}(V)$ . Moreover, with the previous notation we get that:

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} e^{t\xi}.(\bar{a} \otimes_U x) &= \overline{[\tau(\xi), a]} \otimes_U x = \overline{\tau(\xi)a} \otimes_U x = \tau(\xi).(\bar{a} \otimes_U x), \\ \frac{d}{dt}\bigg|_{t=0} e^{t\zeta}.(\bar{a} \otimes_U x) &= \overline{[\bar{\Theta}, a]} \otimes_U x + \bar{a} \otimes_U \theta.x = \overline{\bar{\Theta}a} \otimes_U x - \overline{a\bar{\Theta}} \otimes_U x + \bar{a} \otimes_U \theta.x \\ &= \bar{\Theta}.(\bar{a} \otimes_U x) - \bar{a} \otimes_U \theta.x + \bar{a} \otimes_U \theta.x \\ &= \bar{\Theta}.(\bar{a} \otimes_U x) = \tau(\zeta).(\bar{a} \otimes_U x). \end{aligned}$$

This shows that  $\Psi N \in \text{mod}^{G \times C}(\mathcal{D}_V)$ . The equality  $(\Psi N)^G = \bar{1} \otimes_U N$  follows easily from the definition of the  $G$ -action on  $\Psi N$ , hence  $\Phi \Psi N = N$ .

(2) Note that there is a surjective  $(G \times C)$ -equivariant morphism of  $\mathcal{D}(V)$ -modules  $\mathfrak{m} : \Psi \Phi M = \mathcal{N} \otimes_U M^G \twoheadrightarrow M$  given by  $\mathfrak{m}(\bar{a} \otimes_U x) = a.x$ . Set  $L = \text{Ker } \mathfrak{m}$ , hence  $\mathfrak{m} : (\Psi \Phi M)/L \xrightarrow{\sim} M$ . Then  $L \in \text{mod}^{G \times C}(\mathcal{D}_V)$  is generated by  $L^G$ , by hypothesis, and we obtain ( $G$  is reductive):

$$M^G \cong (\Psi \Phi M/L)^G = (\Psi \Phi M)^G/L^G = (\Phi \Psi \Phi M)/L^G = M^G/L^G.$$

This implies  $L^G = (0)$ , thus  $L = 0$  and  $\Psi \Phi M \equiv M$ . The last statement follows from Lemma 5.15  $\square$

The previous proposition and work of P. Nang lead to the following:

**Conjecture 5.17.** Let  $(\tilde{G} : V)$  be of Capelli type with  $\dim V//G = 1$ . Then the categories  $\text{mod}^{G \times C}(\mathcal{D}_V)$  and  $\text{mod}^\theta(U)$  are equivalent via the functors  $\Phi$  and  $\Psi$ .

By Proposition 5.16, this conjecture is equivalent to showing that  $M = \mathcal{D}(V)M^G$  for all  $M \in \text{mod}^{G \times C}(\mathcal{D}_V)$ .

P. Nang [39, 41, 43] has proved Conjecture 5.17 in the cases:  $(\text{SO}(n) \times \mathbb{C}^* : \mathbb{C}^n)$ ,  $(\text{GL}(n) \times \text{SL}(n) : \text{M}_n(\mathbb{C}))$ ,  $(\text{GL}(2m) : \bigwedge^2 \mathbb{C}^{2m})$ . It would be interesting to obtain a uniform proof in the eight cases where  $(\tilde{G} : V)$  is of Capelli type and  $\dim V//G = 1$  (see Appendix A). As observed above the category  $\text{mod}^\theta(U)$  has a nice combinatorial description, which would give, when  $G$  is simply connected, a

classification of the regular holonomic modules on  $V$  whose characteristic variety is contained in  $\tilde{\mathcal{C}}(V)$ .

## APPENDIX A. IRREDUCIBLE MF REPRESENTATIONS

	$(\tilde{G} : V)$	$\deg f$	$b(s)$	Capelli	com. parabolic
(1)	$(\mathrm{SO}(n) \times \mathbb{C}^* : \mathbb{C}^n)$	2	$(s+1)(s+n/2)$	yes	yes
(2)	$(\mathrm{GL}(n) : S^2 \mathbb{C}^n)$	$n$	$\prod_{i=1}^n (s+(i+1)/2)$	yes	yes
(3)	$(\mathrm{GL}(n) : \bigwedge^2 \mathbb{C}^n), n \text{ even}$	$n/2$	$\prod_{i=1}^{n/2} (s+2i-1)$	yes	yes
(4)	$(\mathrm{GL}(n) \times \mathrm{SL}(n) : \mathrm{M}_n(\mathbb{C}))$	$n$	$\prod_{i=1}^n (s+i)$	yes	yes
(5)	$(\mathrm{Sp}(n) \times \mathrm{GL}(2) : \mathrm{M}_{2n,2}(\mathbb{C}))$	2	$(s+1)(s+2n)$	yes	no
(6)	$(\mathrm{SO}(7) \times \mathbb{C}^* : \mathrm{spin} = \mathbb{C}^8)$	2	$(s+1)(s+4)$	yes	no
(7)	$(\mathrm{SO}(9) \times \mathbb{C}^* : \mathrm{spin} = \mathbb{C}^{16})$	2	$(s+1)(s+8)$	no	no
(8)	$(\mathrm{G}_2 \times \mathbb{C}^* : \mathbb{C}^7)$	2	$(s+1)(s+7/2)$	yes	no
(9)	$(\mathrm{E}_6 \times \mathbb{C}^* : \mathbb{C}^{27})$	3	$(s+1)(s+5)(s+9)$	no	yes
(10)	$(\mathrm{GL}(4) \times \mathrm{Sp}(2) : \mathrm{M}_4(\mathbb{C}))$	4	$(s+1)(s+2)(s+3)(s+4)$	yes	yes
(3')	$(\mathrm{GL}(n) : \bigwedge^2 \mathbb{C}^n), n \text{ odd}$	—	—	yes	no
(4')	$(\mathrm{GL}(n) \times \mathrm{SL}(m) : \mathrm{M}_{n,m}(\mathbb{C})), n \neq m$	—	—	yes	no
(11)	$(\mathrm{Sp}(n) \times \mathrm{GL}(1) : \mathbb{C}^{2n})$	—	—	yes	no
(12)	$(\mathrm{Sp}(n) \times \mathrm{GL}(3) : \mathrm{M}_{2n,3}(\mathbb{C}))$	—	—	no	no
(10')	$(\mathrm{GL}(n) \times \mathrm{Sp}(2) : \mathrm{M}_{n,4}(\mathbb{C})), n \neq 4$	—	—	yes	no
(13)	$(\mathrm{SO}(10) \times \mathbb{C}^* : \frac{1}{2}\mathrm{spin} = \mathbb{C}^{16})$	—	—	yes	no

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